SPECIAL PROJECT PROGRESS REPORT

All the following mandatory information needs to be provided. The length should *reflect the complexity and duration* of the project.

Reporting year	**2023**…………………………………………………….……
Project Title:	**Role of finite non-gaussianity in the evolution of wind wave fields, with applications to freak wave prediction**
Computer Project Account:	**SPGBSHRI** ……………………………………….……
Principal Investigator(s):	**Prof V.I. Shrira** ………………………………….……
	…………………………………………………………….……
Affiliation:	**School of Computer Science and Mathematics, Keele University, Keele ST5 5BG UK** ……………………..……
Name of ECMWF scientist(s) collaborating to the project (if applicable)	…………………………………………………………….…… …………………………………………………………….……
Start date of the project:	**2022**…………………………………………………….……
Expected end date:	**2024**…………………………………………………….……

Computer resources allocated/used for the current year and the previous one

(if applicable)
Please answer for all project resources

		Previous year		Current year	
		Allocated	Used	Allocated	Used
High Performance Computing Facility	(units)	**1000000**	**900000**	**1000000**	**74000**
Data storage capacity	(Gbytes)	**100**	**100**	**100**	**100**

Summary of project objectives (10 lines max)

1. To create, for the first time, the numerical model of wind waves within the kinetic theory with the account for weak nonlinearity and weak non-**G**aussianity.

2. To perform direct comparisons with the DNS

3. To examine implications for the probability of freak waves

4. To get new insights into the input and dissipation functions

5. To formulate recommendations for wind wave modelling

Summary of problems encountered (10 lines max)

No particular problems encountered……………………………………………………………………

Summary of plans for the continuation of the project (10 lines max)

1. To develop further the numerical model for continuous wave fields, optimize it for large-scale supercomputing, find out the current limits due to memory availability

2. To perform direct comparisons with the DNS results

3. To introduce wind forcing into the model, study its performance for different forcing regimes, compare with the DNS simulations

4. To perform a direct comparison of the evolution of higher statistical moments

List of publications/reports from the project with complete references

Annenkov, S.Y. and Shrira, V.I., 2022. Effects of finite non-Gaussianity on evolution of a random wind wave field. *Physical Review E*, *106*(4), p. L042102.

Annenkov, S.Y. and Shrira, V.I., 2022. New wave kinetic equation with the account for finite non-Gaussianity. The 28th WISE meeting, Brest, France, 29 May – 2 June 2022.

Annenkov, S.Y. and Shrira, V.I. 2023. Effects of finite non-Gaussianity and finite nonlinearity in wave kinetics. The 29th WISE meeting, Princeton University, USA, May 7-10 2023.

Summary of results

If submitted **during the first project year**, please summarise the results achieved during the period from the project start to June of the current year. A few paragraphs might be sufficient. If submitted **during the second project year**, this summary should be more detailed and cover the period from the project start. The length, at most 8 pages, should reflect the complexity of the project. Alternatively, it could be replaced by a short summary plus an existing scientific report on the project attached to this document. If submitted **during the third project year**, please summarise the results achieved during the period from July of the previous year to June of the current year. A few paragraphs might be sufficient.

Motivation

There is a wide consensus that the standard wave kinetics (and its main tool – the Hasselmann kinetic equation, KE) does capture the main features of wave field evolution. At the same time, there are obvious discrepancies between modelled and measured spectral shapes: well-developed wind waves have wider, less peaked spectra than the self-similar spectral shape of the Hasselmann equation solutions.

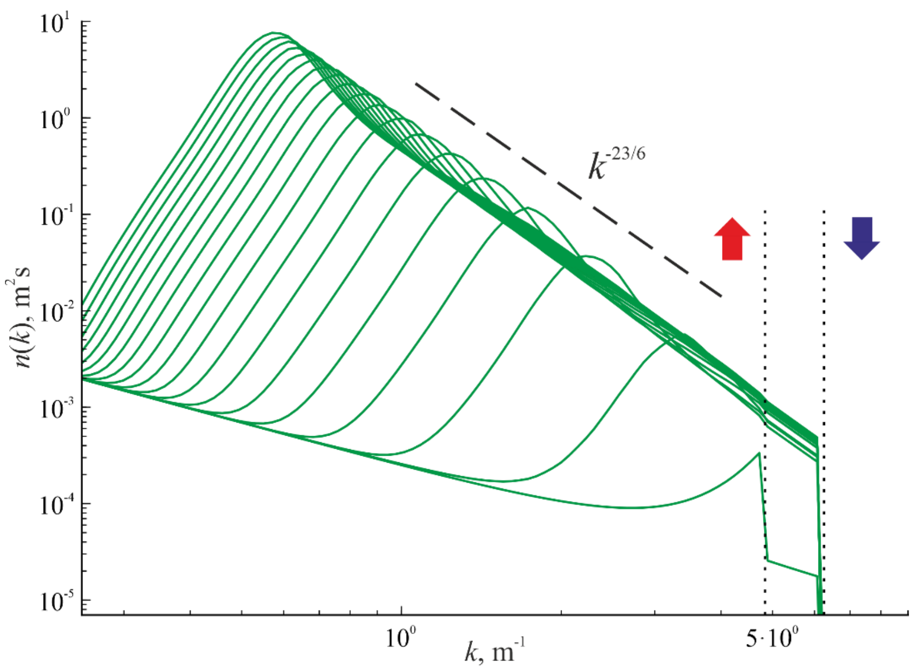


Fig.1. Evolution of a wave field under constant wind forcing *U/c*=2.0, simulated with the Hasselmann equation (WRT algorithm, code kindly provided by Gerbrant van Vledder)

The Hasselmann equation predicts (under constant wind forcing and for sufficiently large time) self-similar evolution

$$n(k) = at^{23/11}U(bkt^{6/11})$$

where $U(\xi)$ is the shape function. This shape function is the same regardless of the nonlinearity level, since the equation is homogeneous with respect to spectral densities. The downshift rate has a simple rescaling law (Zakharov et al 2015).

This self-similar shape is not close to the known parameterisations of wind wave spectra, which are not of universal shape.

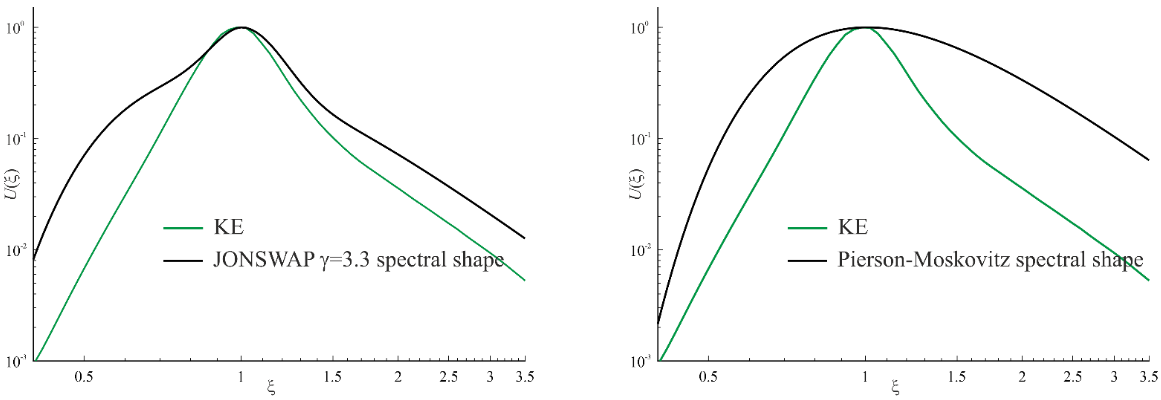


Fig.2. Comparison of the Hasselmann equation solution (KE) and Pierson-Moskovitz spectral shape

Mature wind waves are known to tend to Pierson-Moskovitz spectral shape, which does not have the high spectral peakedness predicted by the Hasselmann equation. Moreover, even younger waves, although more peaked, still do not have spectral shapes close to those numerical predictions. This discrepancy between modelled and observed spectral shapes is a long-standing problem.

Self-similarity laws of spectral development, derived from the Hasselmann equation, do not depend on wind speed (Zakharov et al 2015). The spectral shape is exactly the same for all levels of nonlinearity (even infinitesimal). Moreover, the spectral front is straight, which means that the spectral components grow exponentially almost to the spectral peak. This is, however, in contrast with the expected behaviour of a nonlinear dynamical system.

Insight from direct numerical simulations (DNS)

Some insight can be obtained from direct numerical simulations based on the Zakharov equation (discussed in detail in reports for the previous Special project). First, although the DNS predicts evolution close to that predicted by the KE, the spectra have a different, less peaked shape. Moreover, this spectral shape appears to correspond better to results of measurements, at least for more developed wind wave fields, where the spectral shape should be defined mostly by nonlinearity (Fig.3).

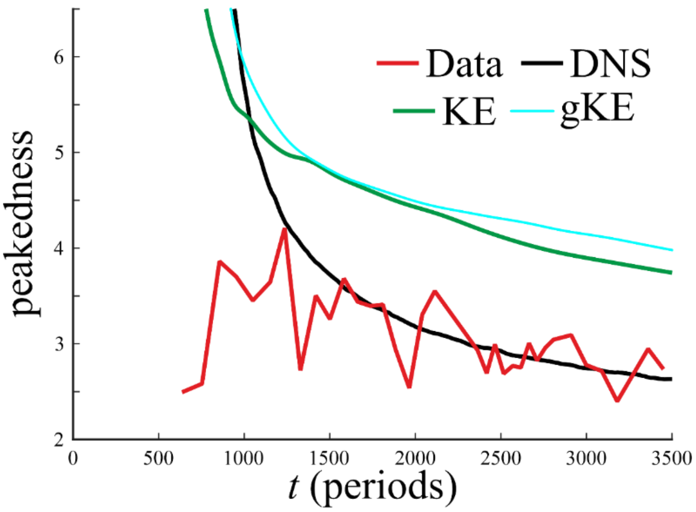


Fig.3. Evolution of spectral peakedness (Goda 1970) under strong wind (HyMeX experiment and DNS simulations). For large times, DNS shows good agreement with measurements. The Hasselmann equation far overestimates the peakedness, as does the generalised kinetic equation (gKE)

Second, the DNS allows to have a close look at the "anatomy" of the spectral growth, which is best seen without angle averaging (for $\theta = 0$ and close angles). If we consider the growth of an initially small harmonic at the bottom of the spectral front, at first the growth is quasi-deterministic. Close directions all grow at the same time, gradually intermittency appears. Then the growth breaks (even with a small drop), and resumes at a slower rate, with increasing intermittency. Close directions get de-synchronised in the peak area. After the second peak, there is still slow random growth (apparently, dependent on breaking parameterisation, with small secondary peaks on the slope). It is interesting that this complicated picture of spectral growth seems to be supported by measurements, at least to some extent (Fig.4). But the KE predicts exponential growth almost to the peak, which is difficult to interpret. Since the amplitude of a growing harmonic eventually becomes comparable with the amplitudes of the harmonics it is interacting with, a qualitative change in the growth regime is to be expected.

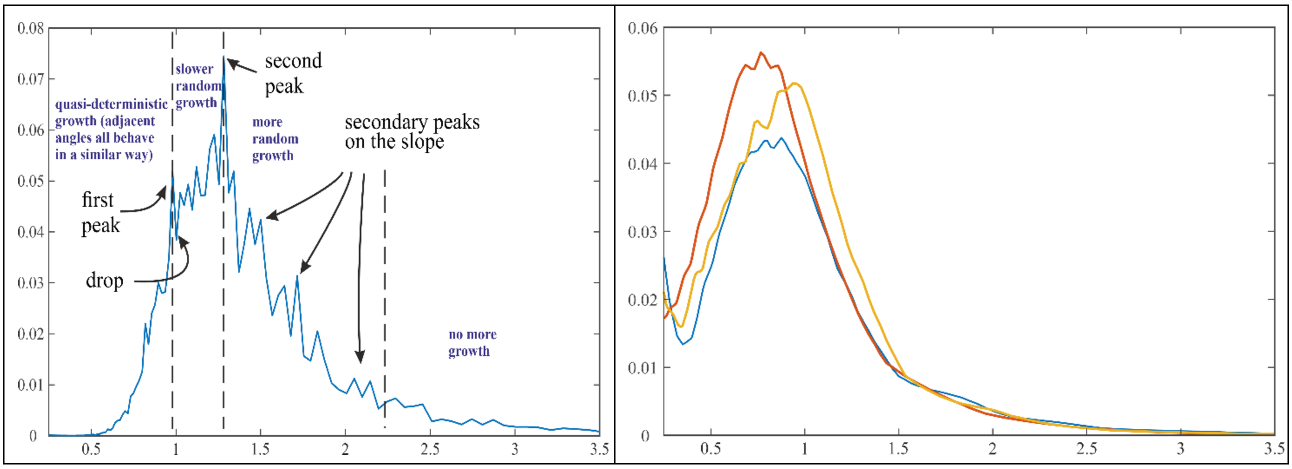


Fig.4. Left panel: Details of the spectral growth obtained by DNS. Spectral shape shown corresponds to zero angle and time about 6700 characteristic periods after the start of wind.
Right panel: Three consecutive non-integrated (zero angle) spectra measured in GOTEX experiment (Romero & Melville 2009).

How can we make the DNS solution approach the KE one? The DNS algorithm is based on the idea of coarse-graining of a wave field (Annenkov & Shrira 2018). When the coarse-graining parameter $\lambda_k = 0$, there are no interactions and a wave field is Gaussian.

When λ_k is increased, the number of wave interactions grows quadratically, the rate of spectral evolution quickly increases and then saturates, at a certain value of λ_k depending on grid resolution. Thus, λ_k can be used to create wave fields with the same level of nonlinearity, but different levels of non-Gaussianity.

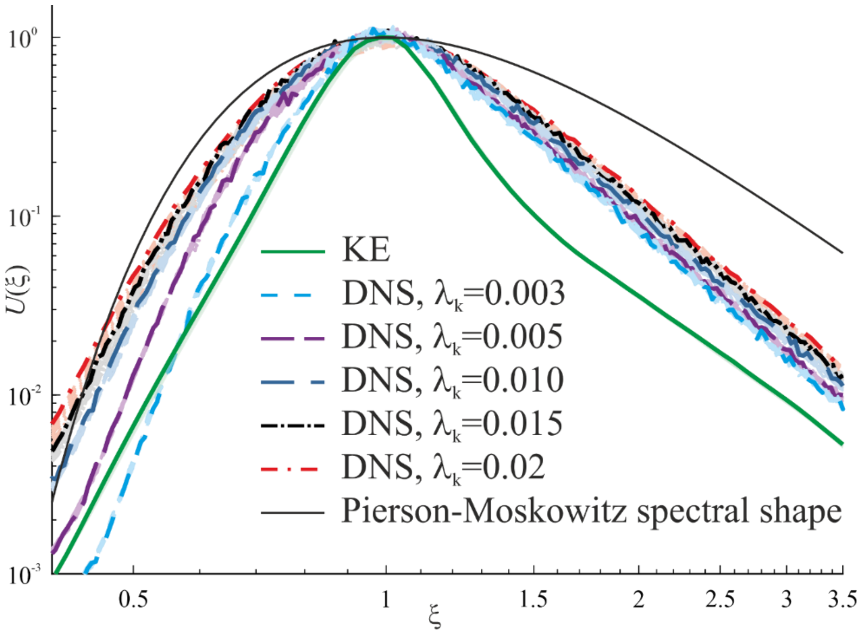


Fig.5. Self-similar shape function $U(\xi)$, $\xi = kt^{6/11}$, extracted from the numerical solutions at the last 1000 wave periods of evolution. Shapes at every 100 periods are shown in light colours, the final curve is in darker colour of the same hue, normalized for $U(1) = 1$.

In the DNS algorithm, the resonance condition is relaxed into $\boldsymbol{k}_0+\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3 = \Delta\boldsymbol{k}$, where $|\Delta\boldsymbol{k}|/k_{min} < \lambda_k \overline{\omega}/\omega_{min}$. For small λ_k the wave field is nearly Gaussian, with very slow evolution, and the shape function is close to the KE spectral shape. With the increase of λ_k the shape function approaches a different form resembling Pierson-Moskowitz spectral shape.

Hasselmann equation and non-Gaussianity

So is appears that origin of the discrepancy is in the non-Gaussianity effects, accounted for by the DNS. But how does the KE treat the non-Gaussianity? The exact theory includes a chain of equations

$$\frac{\partial n_0}{\partial t} = 2\mathrm{Im}\int T_{0123}J_{0123}\delta_{0+1-2-3}d\boldsymbol{k}_{123}.$$

$$\left(\frac{\partial}{\partial t} - \mathrm{i}\Delta\omega\right)J_{0123} = 2\mathrm{i}\int\{T_{0456}\delta_{0+4-5-6}I_{156234}$$
$$+T_{1456}\delta_{1+4-5-6}I_{056234} - T_{2456}\delta_{2+4-5-6}I_{014356} - T_{3456}\delta_{3+4-5-6}I_{014256}\}d\boldsymbol{k}_{456}.$$

Here, n_0 is the 2nd order correlator, $\langle b_0 b_1\rangle = n_0\delta_{0-1}$, J_{0123} is the 4th order cumulant, I_{ijkmnl} are 6th order correlators. To close the system, we neglect the 6th order cumulant, and then I_{ijkmnl} can be expressed in terms of lower-order correlators, i.e. in terms of n_j (second-order correlators) and J_{0123} (fourth order cumulants).

If we keep the theory exact (that is, we neglect only the 6th order cumulant and keep the 4th order ones), the full equations are

$$\frac{\partial n_0}{\partial t} = 2\mathrm{Im}\int T_{0123}J_{0123}\delta_{0+1-2-3}\mathrm{d}\boldsymbol{k}_{123}, \qquad (1)$$

$$\left(\frac{\partial}{\partial t} - \mathrm{i}\Delta\omega\right)J_{0123} = 2\mathrm{i}T_{0123}f_{0123} + 2\mathrm{i}\left(\hat{L}J\right)_{0123}, \qquad (2)$$

where $\hat{L}J$ is a linear operator

$$\left(\hat{L}J\right)_{0123} = M_{0123} + M_{1023} - M_{2301} - M_{3201},$$

$$M_{0123} = n_1\int T_{0145}J_{4523}\delta_{0145}\mathrm{d}\boldsymbol{k}_{45} - n_2\int T_{0425}J_{1534}\delta_{0425}\mathrm{d}\boldsymbol{k}_{45} - n_3\int T_{0435}J_{1524}\delta_{0435}\mathrm{d}\boldsymbol{k}_{45}.$$

Operator $\hat{L}J$ describes the interaction between different 4th order cumulants. While wave harmonics interact as quartets, cumulants interact as triplets. Examples are: $\mathbf{k}_0 + \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3 = \mathbf{k}_4 + \mathbf{k}_5$, or $\mathbf{k}_0 - \mathbf{k}_1 = \mathbf{k}_2 - \mathbf{k}_3 = \mathbf{k}_4 - \mathbf{k}_5$). This system of equations is exact within the assumptions underlying the wave turbulence theory.

The standard kinetic theory is obtained by neglecting $\hat{L}J$. Then the equation for J_{0123} can be solved as

$$J_{0123}(t) = 2\mathrm{i}T_{0123}\int_0^t e^{-\mathrm{i}\Delta\omega(\tau-t)}f_{0123}\mathrm{d}\tau + J_{0123}(0)e^{\mathrm{i}\Delta\omega t} \qquad (3)$$

(Annenkov & Shrira 2006), and the equation for n_0 becomes

$$\frac{\partial n_0}{\partial t} = 4\mathrm{Re}\int\left\{T_{0123}^2\left[\int_0^t e^{-\mathrm{i}\Delta\omega(\tau-t)}f_{0123}\mathrm{d}\tau\right] - \frac{\mathrm{i}}{2}T_{0123}J_{0123}(0)e^{\Delta\omega t}\right\}\delta_{0+1-2-3}\mathrm{d}\boldsymbol{k}_{123},$$

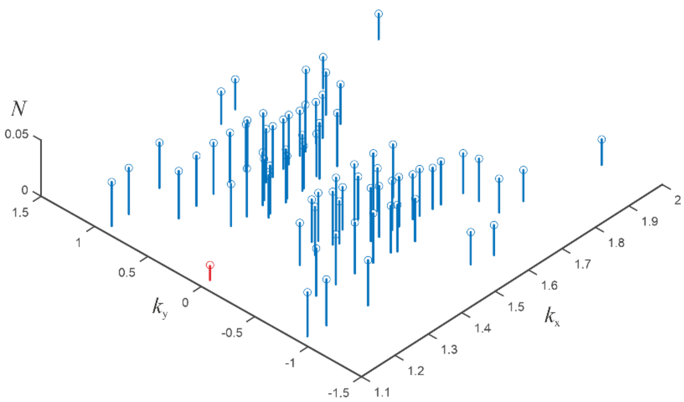
which is the generalised kinetic equation. Taking the large-time limit, we recover the Hasselmann equation

$$\frac{\partial n_0}{\partial t} = 4\pi \int T_{0123}^2 \, f_{0123} \delta_{0+1-2-3} \delta(\Delta\omega) \mathrm{d}\mathbf{k}_{0123}.$$

But can the operator $\hat{L}J$ be safely dropped? The answer is not clear *a priori*. The operator represents the next-order correction, but as it stands on the right-hand side of the evolution equation for correlators, in the long term it may well play a role. Moreover, in that equation, we have spectral amplitudes n_j as coefficients. They become much larger around the spectral peak. Strictly speaking, this term is dropped just because the problem with it is far too complicated, and a kinetic equation in a closed form, describing evolution of spectrum in terms of spectrum, cannot be obtained. Meanwhile, neglecting $\hat{L}J$ means omitting the effects of finite non-gaussianity (incidentally, this is seen from the fact that the Hasselmann equation can be derived assuming random phases (Onorato & Dematteis 2020). The problem, however, is that if finite non-Gaussianity is neglected, we cannot hope to take finite nonlinearity effects into account either.

Discrete problem

To get some idea about the significance of finite non-Gaussianity term, it makes sense to consider a simple test problem. We build a *discrete* wave system which mimics the growth of an initially small harmonic on the spectral front. The target is to build a discrete system with a relatively small number of harmonics (dozens), well linked by nonlinear interactions and showing slow evolution of averaged amplitudes, so that the non-Gaussianity remains low. An example of such a system is shown in fig. 6. Evolution (simulated with the Zakharov equation, with averaging over 10000 realisations) is characterised by slow (over thousands of wave periods) growth of the lowest-frequency harmonic.



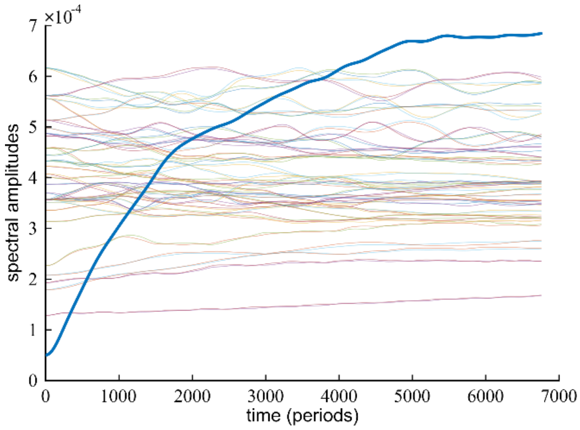


Fig.6. Left panel: Initial condition (spectral amplitudes vs wavevector components) for the discrete system. The lowest-frequency harmonic is shown in red. Right panel: Evolution of amplitudes vs time, with averaging over 10000 realisations. The amplitude of the lowest-frequency harmonic is shown by thick curve

The target is to try and recreate the growth rate using the equation for correlators (2), with and without the non-gaussianity terms. First, we solve the differential equation for correlators (2), and then obtain the growth rates using (1). Results are shown in Fig.7.

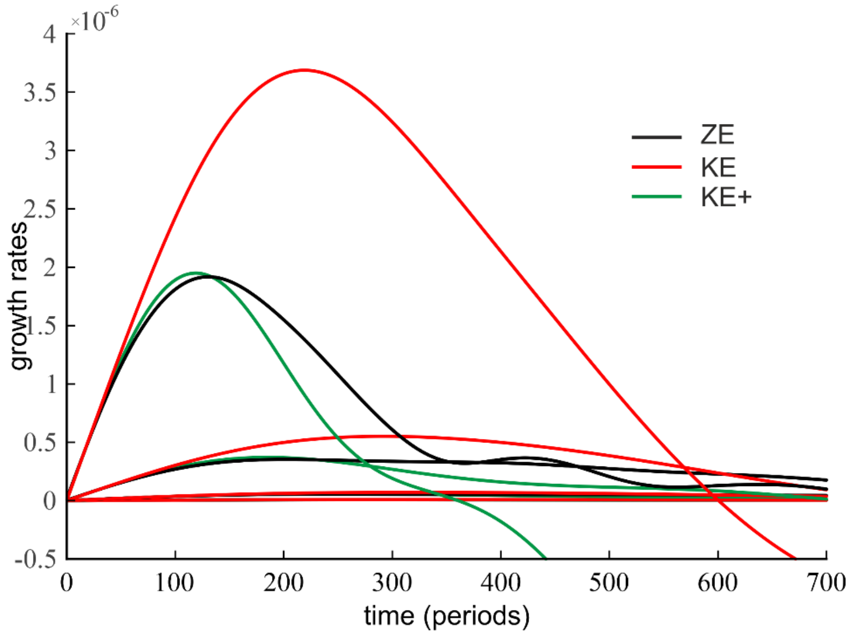


Fig.7. Growth rates for the lowest-frequency harmonic, obtained by numerical solution of Eq. (2) for correlators, using wave amplitudes obtained by numerical solution of the Zakharov equation. Four cases are shown together, for initial wave steepness of the total system equal to 0.035, 0.05, 0.071 and 0.1. Black curve: "true" growth rates from the Zakharov equation. Green curve: solution for the growth rate via correlators, with the account for non-Gaussianity effects. Red curve: solution with non-Gaussianity terms dropped

We plot growth rates for four cases, different only by a multiplier of all initial amplitudes, thus corresponding to different levels of nonlinearity between $\varepsilon = 0.035$ and $\varepsilon = 0.1$, where ε is the total initial wave steepness. Larger initial steepness corresponds to larger absolute growth rates.

Thus, we have found that the always discarded $\hat{L}J$ term is indeed not important for very low nonlinearity only. For larger but still small nonlinearity, the term becomes significant, and for $O(0.1)$ level of nonlinearity, discarding it leads to order 1 error in growth rate. In fact, it turns out that getting a closed kinetic equation for spectra comes at a price. The price is the loss of finite non-Gaussianity effects, even though the Hasselmann equation corresponds to the second-order approximation in nonlinearity. Although non-Gaussianity is weak, neglecting it violates the cornerstone principle of wave turbulence: equal play of weak nonlinearity and weak non-Gaussianity. Ignoring finite non-Gaussianity, we lose finite nonlinearity either. Thus, the analysis of the discrete problem confirms our initial hypothesis about the importance of finite non-Gaussianity effects for wave kinetics.

The most striking and unexpected finding with the discrete problem was the apparent change of scaling. The initial motivation to create the DNS algorithm was to probe the "ε^6 vs ε^4" dilemma: how the dynamical timescale becomes the kinetic one upon ensemble averaging?

We suspected that fast growth rates might acquire faster dynamical scaling due to coherence (large departures from Gaussianity). That would put into question (although locally) the assumptions underlying the wave turbulence theory. That idea was wrong: we could not find large non-Gaussianity, only small one.

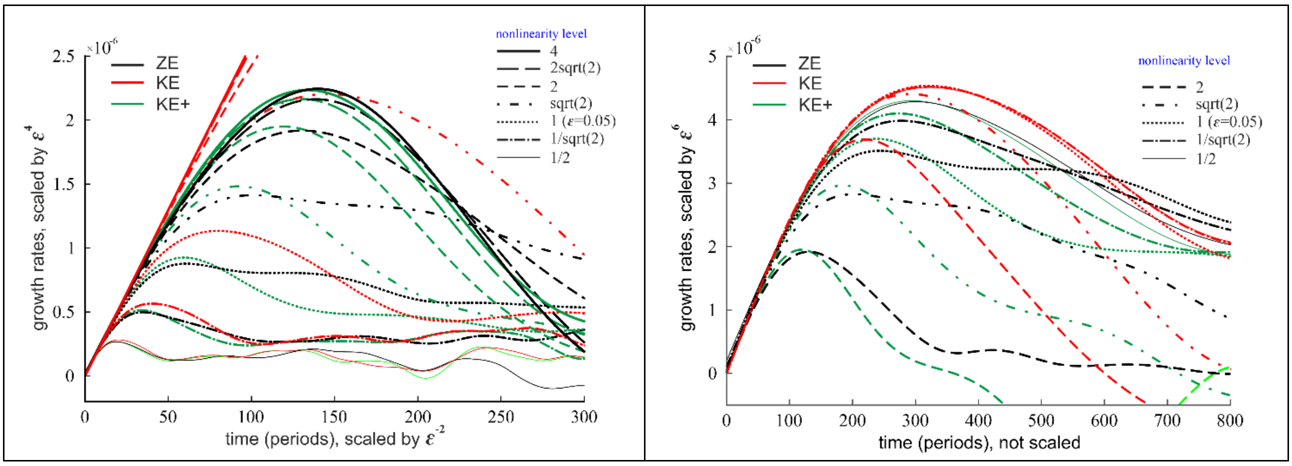


Fig.8. Left panel: growth rates for the discrete problem scaled as ε^4 ("dynamic" scaling). Larger nonlinearity cases have this scaling (while the Hasselmann equation retains strictly kinetic scaling ε^6).
Right panel: same growth rates scaled as ε^6 ("kinetic" scaling). For smaller nonlinearity cases, all curves have this scaling (and all give the same growth rate in the plot scaled accordingly)

Now, as Fig.8 demonstrates, we find the answer: the kinetic scaling of the KE for all nonlinearities appears to be the result of the neglect of non-Gaussianity effects, which starts to play a role above a certain (quite low) nonlinearity threshold.

Algorithm for continuous wave fields

This algorithm is built on the basis of the existing algorithm for the generalised kinetic equation. Here we fully capitalise on the fact that although the gKE algorithm involves many more waves interactions than the standard KE algorithms, the numerical scheme itself is quite simple and transparent, and the non-Gaussianity term can be added with relative ease as an addition to the right-hand side of the equation. The real challenge, however, is that although the time-stepping algorithm can be made rather efficient, it is very demanding in memory, due to the fact that we have to consider interactions of correlators, the number of which is approximately square of the number of degrees of freedom in the system. The efficiency of memory management requires extensive and carefully designed preprocessing, which in turn poses efficiency problems. The algorithm currently undergoes upscaling, and the numerical results will be reported later.

Conclusions so far

In this work, we identify the origin of the discrepancies as the neglect of weak non-Gaussianity (and, hence, weak nonlinearity) in the derivation of the Hasselmann equation.

- We have shown that both the Hasselmann equation and the gKE are derived under more restrictive assumptions than those of the underlying weak turbulence theory
- Getting a kinetic equation for the spectrum in a closed form comes at a price: we have to effectively assume that non-Gaussianity is infinitely small (though it is not the same as assuming that it is zero: then we would have no evolution at all)
- Ignoring finite non-Gaussianity, we lose finite nonlinearity as well, in accordance with the weak turbulence paradigm: equal play of weak nonlinearity and weak non-Gaussianity
- Non-Gaussianity reduces the growth rates when they are large, flattening the peak
- Solutions of the kinetic equation are magnified images of micro-nonlinearity world
- It appears that finite non-Gaussianity effects do not affect integral parameters of spectral evolution, but they do affect spectral shapes, which are increasingly important for new emerging applications of wave modelling
- The gKE, with its transparent perfectly parallel algorithm, appears to be a convenient basis for the algorithm with the account for non-Gaussianity effects