INTERNAL GRAVITY WAVES WITH SHEAR

P.G. Drazin

School of Mathematics
University of Bristol
England

Seminar 1979

1. THE TAYLOR-GOLDSTEIN PROBLEM FOR NORMAL MODES

1.1 Introduction

In these lectures we shall discuss internal gravity waves and instability of parallel flow of a stratified fluid. The theory of these phenomena is the fluid mechanical foundation of much of the dynamical meteorology of mesoscale atmospheric motions (and of dynamical oceanography). Of course we shall make many over-simplifications of the real motion of the atmosphere in order to produce models that are both tractable and instructive. In particular we shall suppose that there is laminar flow of an incompressible inviscid fluid. The assumption that the fluid is incompressible may be relaxed without more than minor technical difficulties when one considers applications to mesoscale motions.

The simplest problem is that of the instability of two layers of fluids with different densities and different horizontal velocities. This is the classic problem of Kelvin-Helmholtz instability. The flow is always unstable unless the velocities of the fluids are equal and the lower fluid is denser than the upper one. Here we shall consider the interplay of the stabilizing influence of gravity on a continuously stratified fluid and of the destabilizing influence of basic shear in a generalized form of the Kelvin-Helmholtz instability, although this generalization is sometimes simply called Kelvin-Helmholtz instability. We shall see that the intuitions that heavy below light fluid is a stabilizing influence, that strong shear is a destabilizing influence, and that light below heavy fluid renders any basic flow unstable are usually correct.

We model the problem by taking a basic state in dynamic equilibrium, with velocity, density and pressure given by

$$\underset{\sim}{u}_* = U_*(z_*)\underset{\sim}{i},\ \rho_* = \bar{\rho}_*(z_*),\ p_* = p_{0*} - g\int^{z_*} \bar{\rho}_*(z_*')\,dz_*' \quad \text{for } z_{1*} \leq z_* \leq z_{2*} \tag{1}$$

respectively, where z_* is the height and g the acceleration due to gravity and where each of the horizontal planes at $z_* = z_{1*}$ and z_{2*} is taken to be rigid. We take scales L of length and V of velocity characteristic of the basic velocity distribution $U_*(z_*)$ and ρ_0 characteristic of the basic density $\rho_*(z_*)$. We assume that the fluid is inviscid and incompressible, density being convected but not diffused. Then the equations of motion, incompressibility and continuity in dimensionless form give

$$\left.\begin{aligned} \rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u}\right) &= -\nabla p - F^{-2}\rho\underset{\sim}{k}, \\ \nabla\cdot\underset{\sim}{u} &= 0, \\ \text{and} \quad \frac{\partial\rho}{\partial t} + \underset{\sim}{u}\cdot\nabla\rho &= 0\,; \end{aligned}\right\} \tag{2}$$

where the <u>Froude number</u> is defined by

$$F = V/\sqrt{gL}, \tag{3}$$

and

$$\rho = \rho_*/\rho_0,\quad \underset{\sim}{u} = \underset{\sim}{u}_*/V \quad \text{etc.}$$

It can be verified that the basic state satisfies the equations for arbitrary distributions $U(z)$, $\bar{\rho}(z)$. To study its stability we put

$$\underset{\sim}{u}(\underset{\sim}{x},t) = U(z)\underset{\sim}{i} + \underset{\sim}{u}'(\underset{\sim}{x},t), \quad p(\underset{\sim}{x},t) = p_o - F^{-2}\int^{z} \bar{\rho}(z')dz' + p'(\underset{\sim}{x},t),$$
$$\rho(\underset{\sim}{x},t) = \bar{\rho}(z) + \rho'(\underset{\sim}{x},t), \tag{4}$$

substitute these expressions into equations (2), and neglect quadratic terms in the small primed quantities to derive the linearized equations for the disturbance. We also take normal modes of the form

$$(\underset{\sim}{u}', p', \rho') = (\hat{\underset{\sim}{u}}(z), \hat{p}(z), \hat{\rho}(z))\exp\{i(\alpha x + \beta y - \alpha ct)\} . \tag{5}$$

Thus equations (2) give

$$\left.\begin{aligned} i\alpha\bar{\rho}(U-c)\hat{u}+\bar{\rho}U'\hat{w} &= -i\alpha\hat{p}, \\ i\alpha\bar{\rho}(U-c)\hat{v} &= -i\beta\hat{p}, \\ i\alpha\bar{\rho}(U-c)\hat{w} &= -D\hat{p} - F^{-2}\hat{\rho}, \\ i\alpha\hat{u} + i\beta\hat{v} + D\hat{w} &= 0, \\ i\alpha(U-c)\hat{\rho} + \bar{\rho}'\hat{w} &= 0, \end{aligned}\right\} \tag{6}$$

where differentiation with respect to z of a basic quantity is denoted by a prime and of a perturbed quantity by D. One may eliminate $\hat{u}$ and $\hat{v}$ from the first two of equations (6) and from the fourth; then one may eliminate $\hat{p}$ and $\hat{\rho}$ in turn with the aid of the third and fifth of equations (6) to find

$$(U-c)\{D^2\hat{w} - (\alpha^2 + \beta^2)\hat{w}\} - U''\hat{w} - \frac{(\alpha^2 + \beta^2)\bar{\rho}'}{\alpha^2 F^2(U-c)\bar{\rho}}\hat{w}$$
$$+ \frac{\bar{\rho}'}{\bar{\rho}}\{(U-c)D\hat{w} - U'\hat{w}\} = 0 . \tag{7}$$

The conditions at the rigid boundaries give

$$\hat{w} = 0 \quad \text{at} \quad z = z_1, z_2 . \tag{8}$$

Yih (1955) applied Squire's transformation to this system. It can be seen that the characteristics of two-dimensional waves are simply related to those of three-dimensional ones, for each three-dimensional wave with numbers (α,β) there being a two-dimensional one with the same value of the complex velocity c but with wave-numbers $((\alpha^2 + \beta^2)^{\frac{1}{2}}, 0)$ and Froude number $\alpha F/(\alpha^2 + \beta^2)^{\frac{1}{2}}$. Thus two-dimensional waves effectively have reduced gravity and magnified relative growth rate $(\alpha^2 + \beta^2)^{\frac{1}{2}}c_i$, and they are usually found to be the most unstable waves. For these reasons we shall henceforth consider only two-dimensional waves.

We used F^{-2} as a dimensionless measure of gravity because it was the first one at hand. However, it can now be seen from equation (7) that F^{-2}

arises only in a product with $-\bar{\rho}'/\bar{\rho}$. Further, the physical effects of gravity on the modes will be seen to create internal rather than surface gravity waves (if the upper boundary is rigid). So we shall use the overall Richardson number J, defined as a characteristic value of

$$-\bar{\rho}'/F^2\bar{\rho} = -gL^2 \frac{d\bar{\rho}_*}{dz_*}\Big/\bar{\rho}_* V^2,$$

rather than the Froude number. It is also convenient to define the Brunt-Väisälä frequency (or the buoyancy frequency) N_* by

$$N_*^2(z_*) = -g \frac{d\bar{\rho}_*}{dz_*}\Big/\bar{\rho}_* = JN^2(z) \;.\; V^2/L^2 \;. \tag{9}$$

Thus $JN^2/U'^2 = -g \dfrac{d\bar{\rho}_*}{dz_*}\Big/\bar{\rho}_*\left(\dfrac{dU_*}{dz_*}\right)^2$ is the local Richardson number of the basic flow at each height z_*, and we shall identify $J^{\frac{1}{2}}N$ as the dimensionless frequency of short internal gravity waves in the case for which $U \equiv 0$ and N is constant (see equation (17)).

In many applications of this theory it happens that $\bar{\rho}_*(z_*)$ varies much more slowly with height than $U_*(z_*)$, so that $-\bar{\rho}'/\bar{\rho} \ll 1$, yet J is nonetheless of order of magnitude unity because $F \ll 1$; in this approximation, which resembles the Boussinesq approximation, we neglect the last two terms of equation (7). Thus the effects of variation of density are neglected in the inertia but retained in the buoyancy.

Considering only two-dimensional disturbances, using the Richardson number instead of the Froude number, and neglecting the inertial effects of the variation of density, we can reduce the system (7), (8) to the form

$$(U-c)(D^2-\alpha^2)\phi - U''\phi + JN^2\phi/(U-c) = 0, \tag{10}$$

$$\alpha\phi = 0 \quad \text{at} \quad z = z_1,\ z_2, \tag{11}$$

where

$$u' = \partial\psi'/\partial z, \quad w' = -\partial\psi'/\partial x \tag{12}$$

and

$$\psi' = \phi(z)\exp\{i\alpha(x-ct)\}\;. \tag{13}$$

Equation (10) is called the Taylor-Goldstein equation in honour of its derivation and exploitation by Taylor and Goldstein, although the equation was independently published by Haurwitz in the same year, 1931.

1.2 Internal gravity waves and Rayleigh-Taylor instability

The important special case of internal gravity waves or Rayleigh-Taylor instability arises when

$$U_* \equiv 0 . \tag{14}$$

Of course this is equivalent to the case when U_* has any constant value, by Galilean transformation. Here there is no scale of the basic velocity, so we use dimensional variables, for which the Taylor-Goldstein problem reduces to

$$c_*^2(D_*^2 - \alpha_*^2)\phi + N_*^2(z_*)\phi = 0, \tag{15}$$

$$\alpha_*\phi = 0 \quad \text{at} \quad z_* = z_{1*}, z_{2*} , \tag{16}$$

a problem originally due to Rayleigh (1883).

The problem has no solution in finite terms for a general function $N_*^2(z_*)$, but there are a few simple solutions known for particular functions $N_*^2(z_*)$. For the simplest, we follow Rayleigh and suppose that $\bar{\rho}_* = \rho_0 \exp(-z_*/H)$ so that $N_*^2 = g/H$ is constant, and deduce at once that

$$c_*^2 = \{\alpha_*^2 + n^2\pi^2/(z_{2*} - z_{1*})^2\}^{-1} N_*^2, \quad \phi = \sin\{n\pi(z_*-z_{1*})/z_{2*}-z_{1*})\}$$
$$\text{for} \quad n = 1, 2, \ldots . \tag{17}$$

This gives a discrete spectrum of internal gravity waves, stable or unstable according as N_*^2 is positive or negative, with a complete set of eigenfunctions.

Detailed properties of internal gravity waves defined by the standard Sturm-Liouville problem (15), (16), for both general and particular density distributions, are described in the books by Yih (1965, Chap. 2) and Krauss (1966). They also treat cases when the upper boundary is a free surface, when the inertial terms due to the variation of density are not negligible, and when the fluid is compressible.

Rayleigh (1883) himself proved the outstanding general property, namely that there is instability if and only if light fluid is locally below heavier fluid, i.e. there is instability if and only if N_*^2 is negative somewhere. His proof runs as follows. Multiply equation (15) by the complex conjugate ϕ^* and integrate from z_{1*} to z_{2*} to deduce that

$$c_*^2 \int_{z_{1*}}^{z_{2*}} |D_*\phi|^2 + \alpha_*^2|\phi|^2 dz_* = \int_{z_{1*}}^{z_{2*}} N_*^2|\phi|^2 dz_*, \tag{18}$$

on integration by parts and use of boundary conditions (16). It follows that c_*^2 and therefore ϕ is real, and that c_* is real if $N_*^2>0$ everywhere. Thus there is stability if $N_*^2>0$ everywhere. To prove the converse, Rayleigh noted that the variational principle associated with the Sturm-Liouville problem (15), (16) gives c_*^2 as the minimum of $\int_{z_{1*}}^{z_{2*}} N_*^2 f^2 dz_*$ over the class of functions $f(z_*)$ with square integrable derivatives such that $\int_{z_{1*}}^{z_{2*}} (D_*f)^2 + \alpha_*^2 f^2 dz_* = 1$. It follows at once by the calculus of variations that $c_*^2<0$ if $N_*^2(z_*)<0$ anywhere.

1.3 Instability

The interplay of the effects of basic shear and buoyancy is seen in the eigensolutions of the Taylor-Goldstein problem (10), (11). We shall show how, for a given flow and wave-number, the modes may be divided into five classes, some of which may be empty:

(1) There is a finite class of non-singular unstable modes. These, giving instability, are the most important and have been given most attention in the literature. The class is empty certainly if the local Richardson number is everywhere greater than or equal to a quarter, the flow then being stable to all waves. These modes are in general the modifications by buoyancy of the unstable modes of shear instability but exceptionally buoyancy with $N^2>0$ everywhere may render unstable a wave of given number which is stable when $N^2 \equiv 0$.

(2) The conjugate damped modes form a finite class of non-singular stable modes.

(3) The marginally stable modes form a finite class of singular neutral modes, each having a branch point at its critical layer.

(4) There is a continuous spectrum of singular neutral modes, each having a singularity no worse than a discontinuity at its critical layer.

(5) The internal gravity waves modified by the basic shear form a discrete class of stable modes when $N^2>0$ everywhere. There are similar unstable modes when $N^2<0$ somewhere.

To discuss the properties of these classes of modes, first note that we may take $\alpha \geq 0$ without loss of generality and that to each unstable mode there corresponds a conjugate stable mode.

The essential mechanism of the instability converts the available kinetic energy of relative motion of layers of the basic flow into kinetic energy of the disturbance, overcoming the potential energy needed to raise or lower fluid when $\frac{d\bar{\rho}_*}{dz_*} < 0$ everywhere. Thus shear tends to destabilize and buoyancy to stabilize the flow. To quantify these tendencies, suppose that two neighbouring fluid particles of equal volumes, at heights z_* and $z_* + \delta z_*$, are interchanged. Then the work δW per unit volume needed to overcome gravity and effect this interchange is given by

$$\delta W = -g\delta\bar{\rho}_*\delta z_*;$$

where $\delta\bar{\rho}_* = \frac{d\bar{\rho}_*}{dz_*}\delta z_*$. In order that horizontal momentum is conserved in the interchange, the particle originally at height z_* will plausibly have final velocity $(U_*+k\delta U_*)\underset{\sim}{i}$ and the other particle $(U_*+\{1-k\}\delta U_*)\underset{\sim}{i}$, where $\delta U_* = \frac{dU_*}{dz_*}\delta z_*$ and k is some number between zero and one. Then the kinetic energy δT per unit volume released by the basic flow in this way is given by

$$\delta T = \tfrac{1}{2}\,\bar{\rho}_* U_*^2 + \tfrac{1}{2}(\bar{\rho}_* + \delta\bar{\rho}_*)(U_* + \delta U_*)^2 - \tfrac{1}{2}\,\bar{\rho}_*(U_* + k\delta U_*)^2$$
$$- \tfrac{1}{2}(\bar{\rho}_* + \delta\bar{\rho}_*)(U_* + \{1-k\}\delta U_*)^2$$
$$= k(1-k)\bar{\rho}_*(\delta U_*)^2 + U_*\,\delta U_*\delta\bar{\rho}_*$$
$$\leqslant \tfrac{1}{4}\,\bar{\rho}_*(\delta U_*)^2 + U_*\delta U_*\delta\bar{\rho}_* ,$$

equality holding for $k = \frac{1}{2}$. Now a necessary condition for this interchange, and thus for instability, is that

$$\delta W \leqslant \delta T$$

and therefore that somewhere in the field of flow

$$- g\,\frac{d\,\bar{\rho}_*}{dz_*} \leqslant \tfrac{1}{4}\,\bar{\rho}_*\left(\frac{dU_*}{dz_*}\right)^2 + U_*\,\frac{dU_*}{dz_*}\,\frac{d\bar{\rho}_*}{dz_*} , \tag{19}$$

i.e.

$$- g\,\frac{d\bar{\rho}_*}{dz_*}\Big/\bar{\rho}_*\left(\frac{dU_*}{dz_*}\right)^2 \leqslant \tfrac{1}{4} \tag{20}$$

if the inertial effects of the variation of density are negligible. The essential idea of this argument is due to Richardson, who in 1920 applied it to turbulence. However, it has been recast by Prandtl, Taylor and many others since. The above form of the argument is essentially that of Chandrasekhar (1961, p.491). The argument is heuristic in the sense that only energetics are considered, the detailed kinematics and dynamics of the interchange of the particles being ignored.

A rigorous form of the argument comes on assuming $c_i \neq 0$, defining H by

$$H = \phi/(U-c)^{\frac{1}{2}}, \tag{21}$$

and substituting H for ϕ in the Taylor-Goldstein equation. This gives

$$D\{(U-c)DH\} - \{\alpha^2(U-c) + \tfrac{1}{2}U'' + (\tfrac{1}{4}U'^2 - JN^2)/(U-c)\}\,H = 0 . \tag{22}$$

Multiplying this equation by H^* and integrating, we find

$$\int_{z_1}^{z_2} (U-c)\{|DH|^2 + \alpha^2|H|^2\} + \tfrac{1}{2}\,U''|H|^2 + \frac{\frac{1}{4}U'^2 - JN^2}{U-c}\,|H|^2\,dz = 0 . \tag{23}$$

The imaginary part of this equation gives

$$-c_i \int_{z_1}^{z_2} |DH|^2 + \alpha^2|H|^2 + (JN^2 - \tfrac{1}{4}\,U'^2)|H|^2/|U-c|^2 dz = 0 . \tag{24}$$

Therefore

$$0 > -\int_{z_1}^{z_2} |DH|^2 dz$$

$$= \int_{z_1}^{z_2} \{(JN^2 - \tfrac{1}{4}U'^2) + \alpha^2|U-c|^2\}|H|^2/|U-c|^2\ dz \tag{25}$$

if $c_i \neq 0$. Therefore the local Richardson number satisfies the inequality

$$JN^2/U'^2 < \tfrac{1}{4} \tag{26}$$

somewhere in the field of flow. This is Howard's (1961) general proof of the necessary condition of instability Miles (1961) proved otherwise for a special class of flows. It can be stated in the form that there is stability if the local Richardson number is everywhere greater than or equal to one quarter. Howard (1961) also showed that inequality (25) gives

$$\alpha^2 c_i^2 \leq \max_{z_1 \leq z \leq z_2}(\tfrac{1}{4}U'^2 - JN^2) \ . \tag{27}$$

A similar integral of the Taylor-Goldstein equation can be used to show that instability implies that

$$U'' = 2(U-c_r)JN^2/\{(U-c_r)^2 + c_i^2\} \tag{28}$$

somewhere in the field of flow (Synge 1933). When J=0 this yields Rayleigh's criterion that there is instability of homogeneous fluid only if the basic velocity profile has a point of inflexion. Unfortunately condition (28) for stratified fluid involves the unknowns c_r and c_i, and so does not give a simple criterion like Rayleigh's.

Another simple integral gives Howard's semicircle theorem, that

$$\{c_r - \tfrac{1}{2}(U_{max} + U_{min})\}^2 + c_i^2 < \{\tfrac{1}{2}(U_{max} - U_{min})\}^2 \tag{29}$$

provided that $c_i \neq 0$ and $N^2 \geq 0$ everywhere in the field of flow. The method also shows that if $N^2 \leq 0$ everywhere then no non-singular neutral mode exists; unfortunately this does not imply instability because a continuous spectrum of singular stable modes with velocity c within the range of U(z) may exist (cf. Chimonas 1979).

For any given basic flow the marginally stable modes are modifications of the 's-modes' for the case J=0, although the significance of a point of inflexion is lost when J=0. For illustration take the example

$$U = \tanh z, \quad N^2 = \text{sech}^2 z \quad \text{for} \quad -\infty<z<\infty \ . \tag{30}$$

Note that this Brunt-Väisälä frequency comes from taking $\bar{\rho} = \exp(-\tanh z)$, so that $\rho_\infty = \bar{\rho}(\infty) = 1/e$ and $\bar{\rho}_{-\infty} = \bar{\rho}(-\infty) = e$. Holmboe (cf. Miles 1961) verified that then a neutral eigensolution of the system (10), (11) is given by

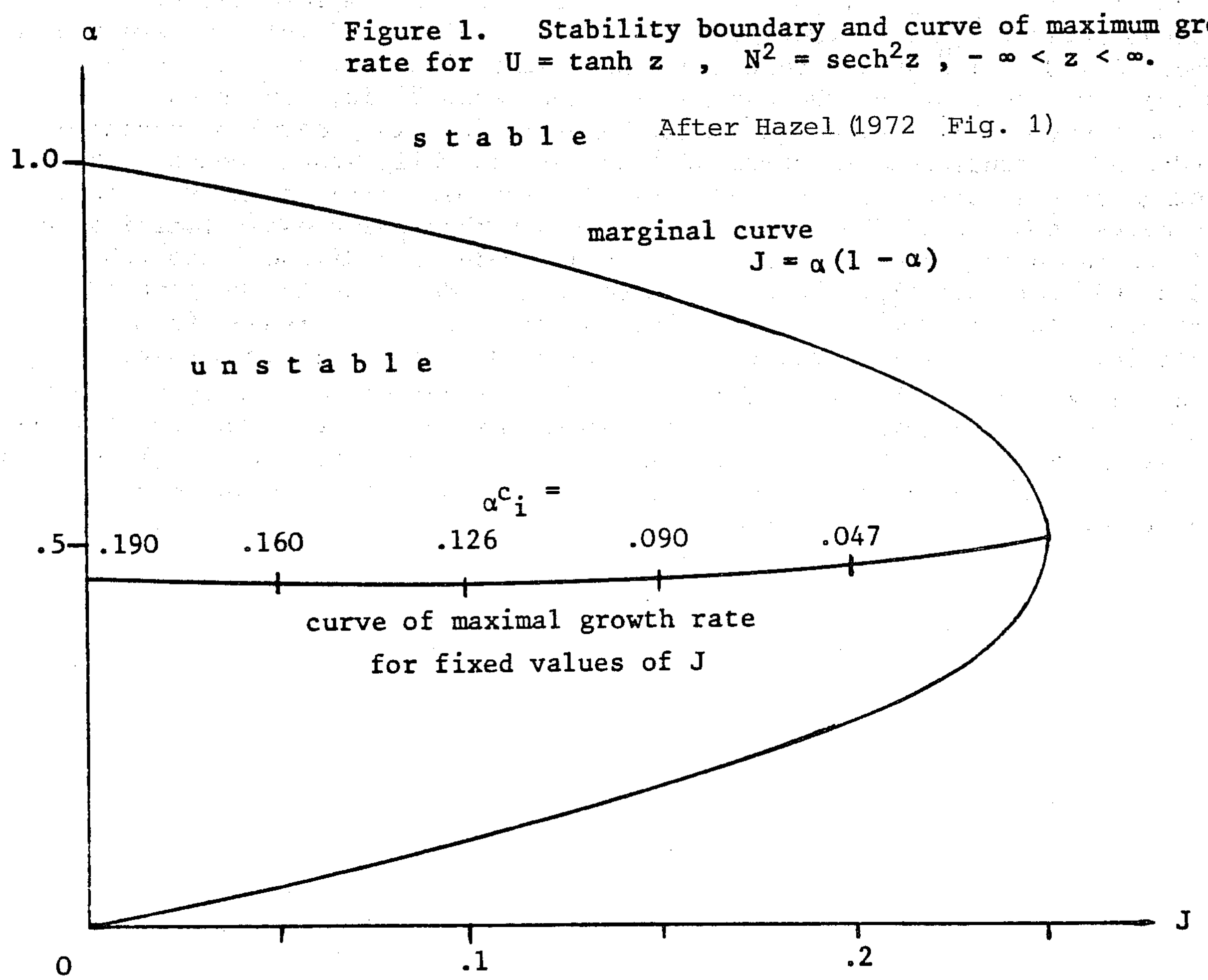
$$c = 0, \quad J = \alpha(1-\alpha), \quad \phi = (\operatorname{sech} z)^{\alpha}(\tanh z)^{1-\alpha} \quad \text{for} \quad 0 \leqslant \alpha \leqslant 1. \tag{31}$$

Miles (1961) examined the branch point at the critical layer where $z = 0$ in the limit as $c_i \downarrow 0$, finding that the solution (31) was the unique marginally unstable solution provided that one interprets $(\tanh z)^{1-\alpha} > 0$ for $z > 0$ but

$$(\tanh z)^{1-\alpha} = e^{-i\pi(1-\alpha)}|\tanh z|^{1-\alpha} \quad \text{for} \quad z<0.$$

Hazel (1972) computed the unstable mode for various values of α and J. These results are shown in Fig. 1. It can be seen that $\alpha c_i \leqslant (\frac{1}{4} - J)^{\frac{1}{2}}$ in accord with inequality (27), equality being attained only when $J = \frac{1}{4}$, $\alpha = \frac{1}{2}$. Thus the condition $J \geqslant \frac{1}{4}$ everywhere happens to be both necessary and sufficient for stability of flows (30). The semi-circle theorem is satisfied, it being found that $c_r = 0$ and $c_i \leqslant 1$ for the unstable mode (with $c_i = 1$ for $\alpha = 0$, $J = 0$). The s-solution can be seen to arise from (31) when $J = 0$ and $\alpha = \alpha_s = 1$.

Figure 1. Stability boundary and curve of maximum growth rate for $U = \tanh z$, $N^2 = \operatorname{sech}^2 z$, $-\infty < z < \infty$.



If U, U' or $\bar{\rho}$ is discontinuous, at $z = z_o$ say, one can show that

$$\Delta[\phi/(U-c)] = 0 ,$$
$$\Delta[\bar{\rho}\{(U-c)D\phi - U'\phi + \phi/F^2(U-c)\}] = 0, \tag{32}$$

in order that the normal velocity and the pressure are continuous at the interface with mean position $z = z_o$.

These conditions are useful to work out simple examples for which the Taylor-Goldstein equation can be solved piecewise in terms of elementary functions.

1.4 Internal gravity waves with basic shear

Next we consider the qualitative character of the eigensolutions of the Taylor-Goldstein problem (10), (11) as J decreases from infinity for fixed functions $U(z)$ and $N(z)$ and a fixed value of α. When $J = \infty$, a condition which we may regard as expressing either $V = 0$ or $g = \infty$, there are internal gravity waves as described in section (b). There we found an infinity of discrete modes with eigenvalues of the form $c = \pm J^{\frac{1}{2}}\gamma_o(n)$ for $n = 1, 2, \ldots,$ and with a complete set of eigengunctions ${}^o\phi_n(z)$; these modes are all stable if $N^2(z) \geq 0$ everywhere. When $J = 0$, which we may regard as expressing either $V = \infty$, $g = 0$ or $N^2(z) \equiv 0$, there arises the case of a homogeneous fluid, for which it is well known that there is a finite number (possibly zero) of unstable modes and a continuous spectrum of singular neutrally stable modes. The change of the pattern of the modes as J decreases from infinity to zero is quite complicated, but briefly one may say that the phase velocities c of the infinite discrete spectrum of stable modes are divided into two classes. For the first class the values of c decrease to the global maximum value of $U(z)$ over the field of flow as J decreases from infinity, and for the second class c increases to the minimum value of $U(z)$ (Banks, Drazin and Zaturska 1976). The eigenfunctions associated with each class form a complete set. As J decreases so much that the local value of the Richardson number is less than a quarter somewhere, either class may be replaced in whole or in part by a finite number of complex eigenvalues or a continuous spectrum of real values.

It can be shown that eigenvalues are given by

$$c = \pm J^{\frac{1}{2}}\gamma_o(n) + O(1) \qquad \text{as} \quad J \uparrow \infty \tag{38}$$

for $n = 1, 2, \ldots$. If z_m is the height of a simple maximum or minimum of $U(z)$ such that $z_1 < z_m < z_2$, then it can be shown that

$$c = U_m - 2JN_m^2/n(n + 2)U_m'' + o(J) \qquad \text{as} \quad J \downarrow 0 \tag{39}$$

for $n = 1,2, \ldots$ and any value of α, where $U_m = U(z_m)$ etc. If, however, the height z_m of the maximum or minimum is at a boundary, say $z_m = z_1$, and the shear does not vanish there, then there is a finite number p (possibly zero) of modes such that

$$U_1 - c \sim U_1'\{A_n(J - J_n)\}^{(1 - 4J_nN_1^2/U_1'^2)^{-\frac{1}{2}}} \qquad \text{as} \quad J \downarrow J_n \tag{40}$$

for $n = 1, 2, \cdots, p$ and some constants $A_n(\alpha)$ and $J_n(\alpha) < \frac{1}{4}U_1'^2/N_1^2$.

For the rest of the modes,

$$U_1 - c \sim B_n \exp\{-n\pi(JN_1^2/U_1'^2 - \tfrac{1}{4})^{-\frac{1}{2}}\} \qquad \text{as} \quad J \downarrow \tfrac{1}{4}U_1'^2/N_1^2 \tag{41}$$

for $n = p + 1, p + 2, \ldots$ and some $B_n(\alpha)$. Here we take $U_1 = U(z_1)$ etc. and may derive similar results if $z_m = U_2$. These results are perhaps most easily understood by seeing figures 2 and 3 for two examples, which also indicate the dependence of c upon n.

1.5 Observational results

Experiments and observations of motions of a stratified fluid are not easy. However, new developments of techniques of remote sensing of the atmosphere by the use of sound, micro-waves and lasers has brought in a new era of observational meteorology. Many of the theoretical and observational results are discussed at length by Gossard and Hooke (1975).

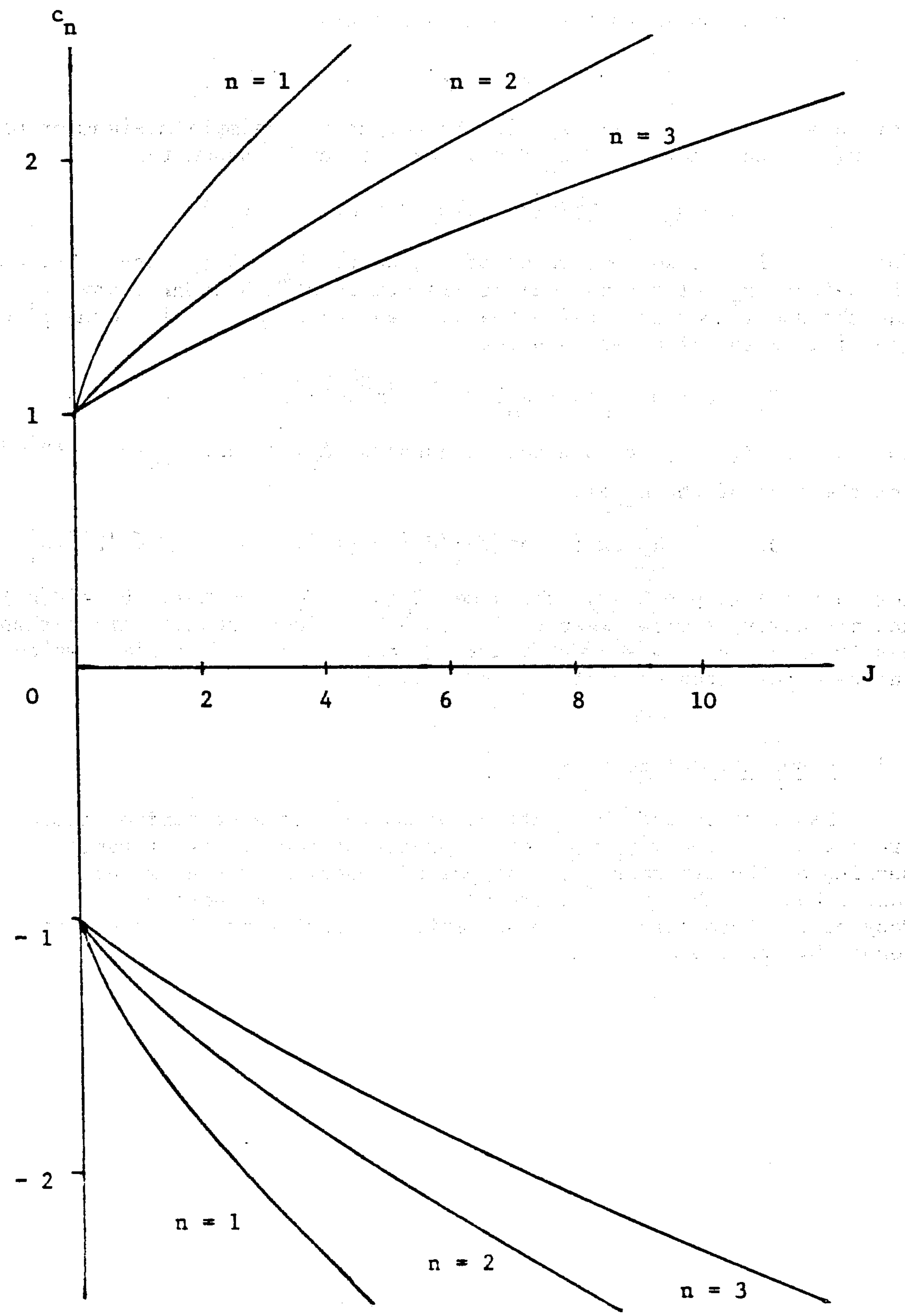


Figure 2 Sinusoidal flow $U = \sin z$, $N^2 = 1$ for $-\pi < z < 1$: c vs J for $n = 1, 2,$ and 3, and $\alpha^2 = \frac{3}{4}$. Note that $U_{max} = 1$ and $U_{min} = -1$. After Banks, Drazin and Zaturska (1976, Fig. 2)

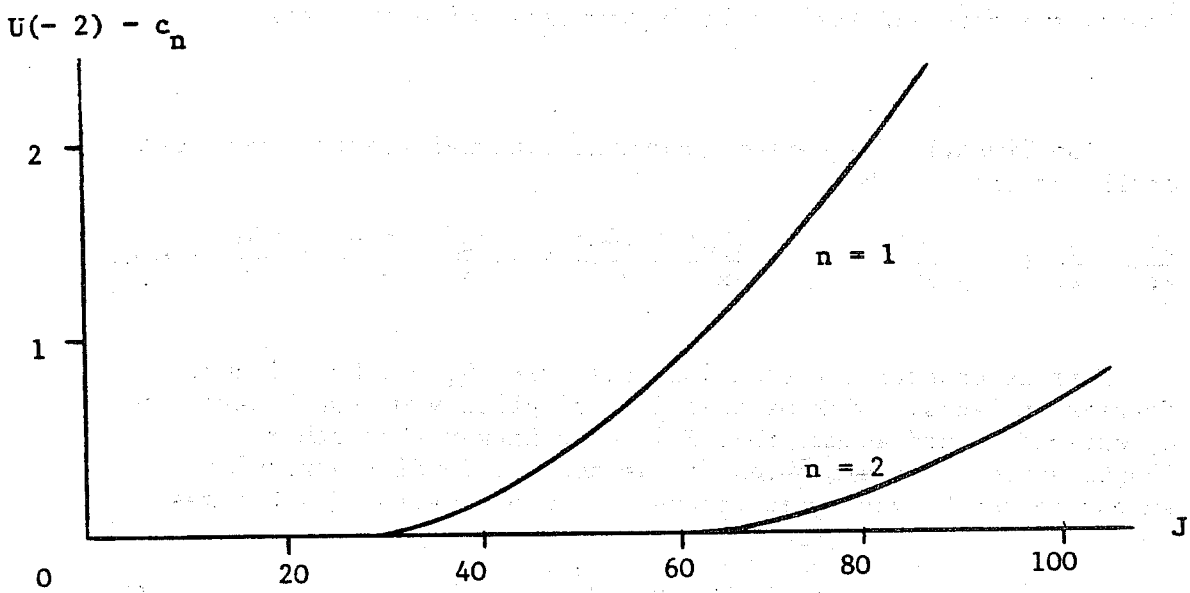


Figure 3. $U = z^3 - z$, $N^2 = 1$ for $-2 < z < 2$: $U(-2) - c_n$ versus J for $n = 1, 2$ and $\alpha = 1$. After Banks, Drazin and Zaturska (1976, Fig. 5).

Note that $J = 25.6$, $p = 1$, and $\mu = 0$ at $J = 30\frac{1}{4}$.

2. THE PROPAGATION OF PLANE INTERNAL GRAVITY WAVES

In Section 1.2 we met internal gravity waves as normal modes; here we shall meet their group velocity, their reflexion, and their production by a source, and shall discuss their relationship to wave fronts and rays and their role in boundary-value problems.

The linearized equation governing internal gravity waves can easily be shown to be

$$\frac{\partial^2}{\partial t^2}\left\{\frac{\partial}{\partial z}\left(\bar\rho\,\frac{\partial w'}{\partial z}\right)+\bar\rho\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial y^2}\right)\right\}-g\,\frac{d\bar\rho}{dz}\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial y^2}\right)=0. \tag{1}$$

This is essentially equation (1.7) with $U_* \equiv 0$ but without Fourier analysis. For mathematical simplicity we shall take $\bar\rho = \rho_o \exp(-z/H)$, and assume that H is much larger than other length scales of the problem (so we neglect density variation except in the buoyancy, as before). Then equation (1) becomes

$$\frac{\partial^2}{\partial t^2}\,\Delta w' + \frac{g}{H}\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial y^2}\right)=0. \tag{2}$$

Equation (2) admits plane wave solutions of the form $w' \propto \exp\{i(\alpha x + \beta y + \gamma z - \omega t)\}$ if the dispersion relation

$$\omega^2 = g(\alpha^2+\beta^2)/H\,\underset{\sim}{\alpha}^2 \tag{3}$$

is satisfied, where the vector wavenumber is given by $\underset{\sim}{\alpha} = (\alpha,\beta,\gamma)$. Note that $\omega^2 \le g/H$, with equality if and only if $\gamma = 0$.

These plane wave solutions also exist when α^2, β^2 or γ^2 is negative, giving external gravity waves which grow or decay exponentially with x, y or z. They occur only if there is an appropriate boundary to ensure that w' is finite; for example, an external wave varying like $\exp(-|\gamma|z)$ above the ground $z = 0$ may occur. Such a wave is said to be external because it is appreciable only near the exterior of the domain of flow.

The conditions of incompressibility and mass conservation give $\mathrm{div}\,\underset{\sim}{u} = 0$ and thence

$$\underset{\sim}{\alpha}\cdot\underset{\sim}{u}' = 0. \tag{4}$$

Thus internal gravity waves are transverse, motion of the fluid being perpendicular to the direction of phase propagation.

The group velocity is given by (cf. Lighthill 1978, Sect.3.6)

$$\underset{\sim}{c}_g = \left(\frac{\partial\omega}{\partial\alpha}, \frac{\partial\omega}{\partial\beta}, \frac{\partial\omega}{\partial\gamma}\right)$$

$$= \frac{\omega\gamma}{(\alpha^2+\beta^2)\underset{\sim}{\alpha}^2}\left(\gamma\alpha, \beta\gamma, -(\alpha^2+\beta^2)\right). \tag{5}$$

Note that $\boldsymbol{\alpha}.\mathbf{c}_g = 0$. It can be shown that the energy flux of the waves is in the direction of the group velocity. Also the phase velocity is given by

$$\mathbf{c} = \omega\boldsymbol{\alpha}/\alpha^2. \tag{6}$$

It follows that

$$\mathbf{c} + \mathbf{c}_g = \frac{\omega}{\alpha^2 + \beta^2} \ (\alpha, \beta, 0).$$

These relations can be represented geometrically in an illuminating way, on taking $\mathbf{c}$ and $\mathbf{c}_g$ to represent segments of a circle on the base of a diameter. This is shown in figure 4. Note that the plane of the circle is vertical and contains the direction of phase propagation of the wave, and that the vertical components of the phase and group velocities are in opposite directions.

To solve a boundary-value problem one is likely to need a real Fourier integral of these complex wave components or wave components in other coordinates, for example cylindrical polars. As a simple example, however, consider waves in the rigid rectangular box $0 \le x \le K$, $0 \le z \le M$. This problem admits eigensolutions

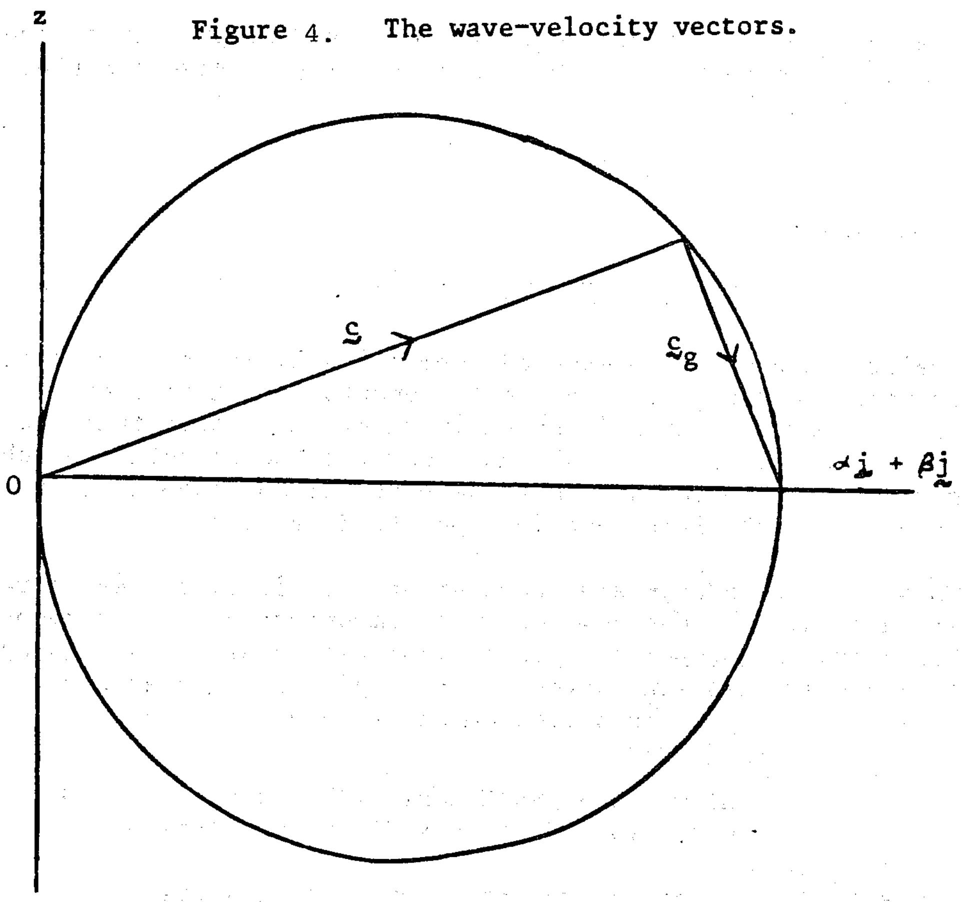
$$\left.\begin{aligned} u' &= A \sin(p\pi x/K)\cos(q\pi z/M)\cos \omega t \\ w' &= A(pM/qK)\cos(p\pi x/K)\sin(q\pi z/M)\cos \omega t \end{aligned}\right\} \tag{7}$$

for p, q = 1, 2, ... and an arbitrary constant A, where

$$\omega^2 = \frac{gp^2}{HK^2} \Big/ \left(\frac{p^2}{K^2} + \frac{q^2}{M^2} \right). \tag{8}$$

This solution represents cellular standing waves. In a crowded room you may sometimes see such waves made visible as a layer of smoke undulates.

Figure 4. The wave-velocity vectors.



3. PROPAGATION OF INTERNAL GRAVITY WAVES WITH BASIC SHEAR

3.1 Airflow over a mountain

An important application of the theory of forced internal gravity waves is to the airflow over a mountain. This application has been treated by many authors (cf. Gossard and Hooke 1975) since the first work of Lyra in 1940. Here, trying to capture the essence of the problem with a minimum of detail, we shall neglect the rotation of the earth, compressibility of air, unsteadiness of flow, nonlinearity, three-dimensionality, and non-hydrostatic effects after Drazin and Su (1975).

We accordingly suppose that $\underset{\sim}{u} \to U(z)\underset{\sim}{i}$, $\rho \to \bar{\rho}(z)$ as $x \to -\infty$ far upstream. Then we put $\underset{\sim}{u} = U(z)\underset{\sim}{i} + \underset{\sim}{u}'(x,z)$ and $\rho = \bar{\rho}(z) + \rho'(x,z)$ as in equation (1.4) and prepare to linearize the equations of motion. It is convenient first to introduce the dependent variable $\zeta(x,z)$, defined as the height of the streamline through the point (x,z) above its level far upstream. (It may help to look at figure 5.) Thus ζ is a Lagrangian vertical displacement such that

$$w' = \frac{D\zeta}{Dt} = U\frac{\partial\zeta}{\partial x} \tag{1}$$

on linearization. The equation of incompressiblity gives

$$0 = \frac{D\rho}{Dt} = U\frac{\partial\rho'}{\partial x} + w'\frac{d\bar{\rho}}{dz} = U\frac{\partial}{\partial x}(\rho' + \zeta\frac{d\bar{\rho}}{dz}).$$

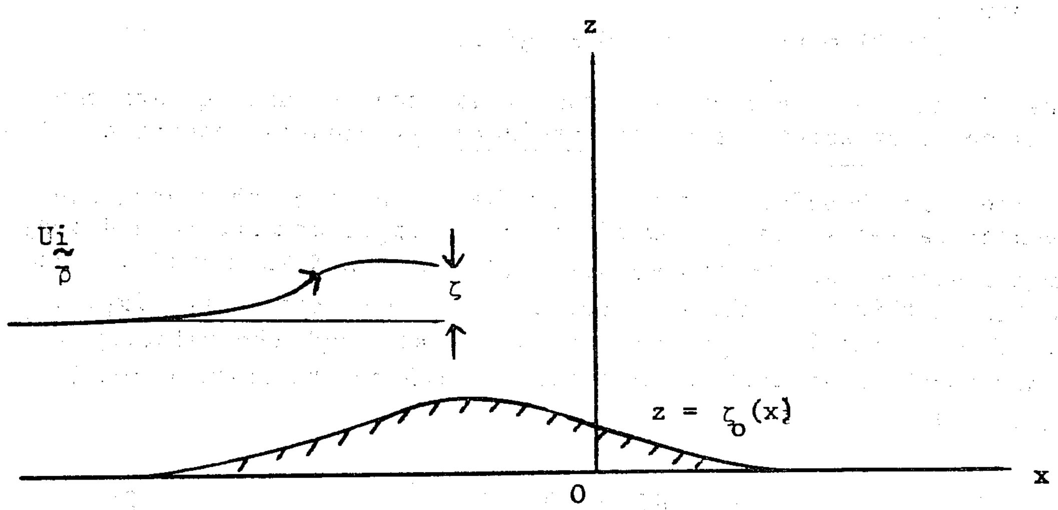


Figure 5. The configuration of the model airflow over a mountain

Therefore, on integration along a streamline, we find

$$\rho' = -\zeta \frac{d\bar{\rho}}{dz} \ . \tag{2}$$

Now mass conservation and hydrostatic balance give

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0, \quad \text{and} \quad \frac{\partial p'}{\partial z} = - g\rho' \tag{3}$$

respectively. Therefore the linearized equation of horizontal momentum gives

$$- \frac{\partial p'}{\partial x} = \bar{\rho}(U \frac{\partial u'}{\partial x} + w' \frac{dU}{dz}) = \bar{\rho}(-U \frac{\partial w'}{\partial z} + w' \frac{dU}{dz})$$

$$= -\bar{\rho}U^2 \frac{\partial}{\partial z} (\frac{w'}{U}) = -\bar{\rho}U^2 \frac{\partial^2\zeta}{\partial z \ \partial x}$$

(if we assume that $U > 0$), and thence

$$p' = \bar{\rho}U^2 \partial\zeta/\partial z. \tag{4}$$

Therefore (2), (3) and (4) give

$$\frac{\partial}{\partial z} (\bar{\rho}U^2 \frac{\partial\zeta}{\partial z}) = g \frac{d\bar{\rho}}{dz} \zeta \ . \tag{5}$$

This is essentially equation (1.7) with $c = 0$, $\beta = 0$, $\alpha^2 << D^2$ and $\zeta \propto \hat{w}/U$; thus the hydrostatic approximation is seen to be equivalent to that of long waves. The general solution is of the form

$$\zeta(x,z) = F(x)\ f(z) + G(x)\ g(z), \tag{6}$$

where F and G are arbitrary functions, and f and g are any two independent solutions of the <u>ordinary</u> differential equation (5).

The upper boundary condition can be shown to be that each wave component radiates energy upwards into the upper atmosphere and away from the source, namely the mountain (Eliassen and Palm 1961). To investigate this we shall suppose, to be both specific and simple, that $U \to U_\infty$ and $\bar{\rho} \sim \rho_0 e^{-z/H}$ as $z \to \infty$, although the velocity of the stratosphere in fact usually varies with height quite strongly. Then we define

$$\gamma = + (g/HU_\infty^2 - \tfrac{1}{4}H^{-2})^{\frac{1}{2}} , \tag{7}$$

supposing that $\gamma^2 > 0$ (which is almost always true in the stratosphere). Then solution (6) gives

$$\bar{\zeta}(\alpha,z) \sim e^{z/2H} \ \{\bar{F}(\alpha)\ e^{i(\alpha x + \gamma z)} + \bar{G}(\alpha)\ e^{i(\alpha x - \gamma z)}\} \text{ as } z \to \infty,$$

where $\bar{\zeta}$, $\bar{F}$ and $\bar{G}$ are the Fourier transforms of ζ, F and G respectively. Now look at the flow in a frame moving with velocity $U_\infty \underset{\sim}{i}$ relative to the mountain. By this Galilean transformation the mountain and the wave appear to move upstream with speed U but the upper atmosphere is reduced to rest, so we may use formula (3.5) with $\omega = -\alpha U_\infty$. That gives us the group velocity with a positive vertical component, and hence upward propagation of energy, if and only if $\gamma\alpha\, U_\infty > 0$. That implies here, where we choose U_∞ and γ to be positive, that

$$\bar{\zeta} \sim \begin{cases} \bar{F}(\alpha) e^{z/2H + i(\alpha x + \gamma z)} & \text{for } \alpha > 0 \\ \bar{G}(\alpha) e^{z/2H + i(\alpha x - \gamma z)} & \text{for } \alpha < 0 \end{cases} \quad \text{as } z \to \infty. \tag{9}$$

The boundary condition that the mountain, with equation $z = \zeta_0(x)$, say, is a streamline gives

$$\zeta(x, 0) = \zeta_0(x). \tag{10}$$

Putting together (9) and (10) with Fourier analysis, we find

$$\zeta = \frac{f(z)}{f(0)} \int_0^\infty \bar{\zeta}_0(\alpha) e^{i\alpha x}\, d\alpha + \frac{f^*(z)}{f^*(0)} \int_{-\infty}^0 \bar{\zeta}_0(\alpha) e^{i\alpha x} d\alpha\,, \tag{11}$$

where $f(z)$ is the solution of equation (5) which behaves like $e^{(i\gamma + 1/2H)z}$ as $z \to \infty$ and $\bar{\zeta}_0$ is the Fourier transform of ζ_0. It follows that

$$\zeta = \mathrm{Re}\left\{\frac{f(z)}{f(0)}\right\} \int_{-\infty}^\infty \bar{\zeta}_0(\alpha) e^{i\alpha x} d\alpha + i\,\mathrm{Im}\left\{\frac{f(z)}{f(0)}\right\} \left(\int_0^\infty - \int_{-\infty}^0\right) \bar{\zeta}_0(\alpha) e^{i\alpha x} d\alpha$$

$$= \zeta_0(x)\ \mathrm{Re}\left\{\frac{f(z)}{f(0)}\right\} + \pi^{-1} P\int_{-\infty}^\infty \frac{\zeta_0(t)}{t-x}\, dt\ \ \mathrm{Im}\left\{\frac{f(z)}{f(0)}\right\}. \tag{12}$$

The Cauchy principal part of the integral in the latter term may be recognised as the Hilbert transform of $\pi\zeta_0$ with many well known properties (see, e.g., Titchmarsh 1948).

To illustrate this theory we take one simple example with

$$\bar{\rho} = \rho_0\, e^{-z/H}, \quad U = \text{constant} \quad \text{for } z \geq 0, \tag{13}$$

and

$$\zeta_0(x) = b^2 h/(b^2 + x^2) \quad \text{for } -\infty < x < \infty. \tag{14}$$

Then it can be shown that the Hilbert transform of ζ_0 is given by

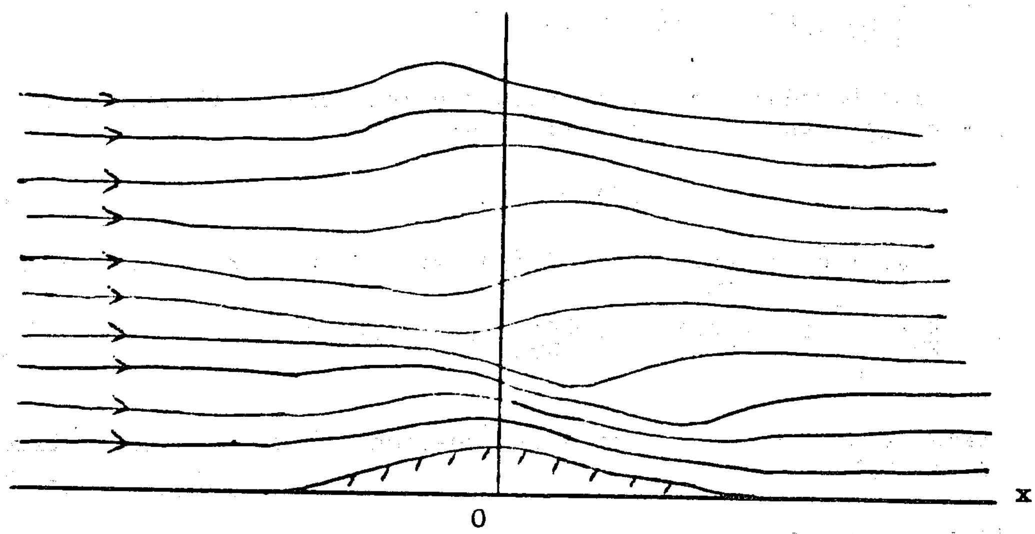
$$\pi^{-1} P\int_{-\infty}^\infty \frac{\zeta_0(t)}{t-x}\, dt = -bhx/(b^2 + x^2) \tag{15}$$

and thence that

$$\zeta(x,z) = bhe^{z/2H}(b \cos\gamma z - x \sin\gamma z)/(b^2 + x^2) . \tag{16}$$

The streamlines of this flow are illustrated in figure 6. Note that the crests of the waves tilt upstream as one rises.

Figure 6. Sketch of typical streamlines for airflow with $U = U_\infty$, $\bar{\rho} = \rho_o e^{-z/H}$, $\zeta_o = b^2h/(b^2 + x^2)$.



3.2 Energy and momentum

Hitherto we have mentioned energy only in asserting that the energy of waves is propagated with the group velocity when the fluid is in a basic state of rest, but there is more to be said.

The linearized equations of motion (2)-(5) give

$$\frac{DE}{Dt} = U\frac{\partial E}{\partial x} = \frac{\partial W_x}{\partial x} + \frac{\partial W_z}{\partial z} - \bar{\rho}\frac{dU}{dz}u'w', \tag{17}$$

where we define

$$E = \tfrac{1}{2}(\bar{\rho}\, u'^2 - g\frac{d\bar{\rho}}{dz}\zeta^2),\ W_x = p'u' \text{ and } W_z = p'w' . \tag{18}$$

We identify E as the energy density of the waves, $\frac{1}{2}\bar{\rho}u'^2$ being the kinetic energy (note that we neglect the kinetic energy of the vertical motion in order to be consistent with the hydrostatic approximation) and $-\frac{1}{2}g\frac{d\bar{\rho}}{dz}\zeta^2$ being the potential energy. Similarly we identify W_x and W_z as the components of the energy flux and $-\bar{\rho}\frac{dU}{dz}u'w'$ as the rate of transfer of energy density from the basic shear flow (this is essentially a Reynolds stress).

Also the vertical flux of horizontal momentum is given by

$$M_x = \bar{\rho}u'w' = -W_z/U. \tag{19}$$

This is related to the drag exerted on the mountain by the wind,

$$D = \int_{-\infty}^{\infty} [p']_{z=0} \frac{\partial \zeta_0}{\partial x} dx = -\int_{-\infty}^{\infty} \left[\bar{\rho}\ U^2 \frac{\partial \zeta}{\partial z}\right]_{z=0} \frac{\partial \zeta_0}{\partial x} dx$$

$$= \frac{1}{U(0)} \int_{-\infty}^{\infty} [p'w']_{z=0}\ dx\ , \tag{20}$$

on resolving the force due to the pressure perturbation in the horizontal direction, because the x-integral of M_x must be independent of height to conserve momentum. For our example (16) we find

$$D = \tfrac{1}{4}\pi\gamma h^2\ \rho_0 U^2. \tag{21}$$

More insight into these properties can be gained from the model of Section 2, with $U \equiv 0$ and negligible inertial effects of density variation. It gives waves with dispersion relation (2.3) and

$$\alpha v' = \beta u',\ w' = -(\alpha^2 + \beta^2)u'/\gamma\alpha,\quad p' = \omega\bar{\rho}u'/\alpha\ . \tag{22}$$

The energy density is then

$$E = \tfrac{1}{2}\bar{\rho}(\underset{\sim}{u}'^2 + N^2 w'^2/\omega^2)$$

$$= \tfrac{1}{2}\bar{\rho}(u'^2 + v'^2 + w'^2 + gw'^2/H\omega^2) \tag{23}$$

$$= \bar{\rho}(\alpha^2 + \beta^2)\underset{\sim}{\alpha}^2\ u'^2/\gamma^2\ \alpha^2\ ,$$

and the wave flux vector

$$\underset{\sim}{W} = (W_x,\ W_y,\ W_z) = (p'u',\ p'v',\ p'w')$$

$$= E\underset{\sim}{c}_g\ , \tag{24}$$

as we asserted earlier. One can also show that $DE/Dt = \mathrm{div}\underset{\sim}{W}$.

Going further back to the model of the Taylor-Goldstein equation (1.10), we find that the average vertical flux of horizontal momentum is given by

$$\bar{M}_x = \bar{\rho}\ \overline{u'w'} = \frac{\alpha\bar{\rho}}{2\pi} \int_0^{2\pi/\alpha} u'w'\ dx$$

$$= \frac{\alpha\bar{\rho}}{2\pi} \int_0^{2\pi/\alpha} \mathrm{Re}\left\{D\phi\ e^{i\alpha(x-ct)}\right\}\ \mathrm{Re}\left\{-i\alpha\phi e^{i\alpha(x-ct)}\right\} dx$$

$$= \tfrac{1}{4}i\alpha\bar{\rho}(\phi^* D\phi - \phi D\phi^*)e^{2\alpha c_i t}\ , \tag{25}$$

and of vertical momentum by

$$\bar{W}_z = \frac{\alpha}{2\pi}\int_0^{2\pi} p'w' \, dx$$

$$= -\tfrac{1}{4}\,\alpha\bar{\rho}\,\{i(U-c_r)(\phi^*D\phi - \phi D\phi^*) + c_i D|\phi|^2\}. \qquad (26)$$

When $c_i = 0$ we deduce that

$$\bar{W}_z = -(U-c)\bar{M}_x\,, \qquad (27)$$

in agreement with (19). Also the Taylor-Goldstein equation gives

$$\frac{d\bar{M}_x}{dz} = \tfrac{1}{2}\alpha\, c_i\, \bar{\rho}\left\{\frac{JN^2(U-c_r)}{|U-c|^4} - \frac{U''}{|U-c|^2}\right\}|\phi|^2, \qquad (28)$$

as in the proof of (1.28). This also shows that if $c_i = 0$ then $\bar{M}_x$ is constant except possibly where $U = c$, and

$$\frac{d\bar{W}_z}{dz} = -U'\,\bar{M}_x\,. \qquad (29)$$

3.3 Critical layers

Hitherto we have shunned the singularity of the Taylor-Goldstein equation at its critical layer, say $z = z_c$ where $U(z) = c$. This singularity in fact gives rise to the continuous spectrum of neutral modes and to the interpretation of the branch - points of eigenfunctions of marginally stable modes, both of which are mentioned briefly in Section 1. Also in this Section we implicitly assumed that $U \neq 0$ in order that our theory of lee waves would be nonsingular. But what happens if $U = 0$?

One may treat a problem of wave propagation after Booker & Bretherton (1967) to illustrate both the mathematical ideas and physical importance of the critical layer. We consider a monochromatic two-dimensional internal gravity wave of numbers α and $\beta = 0$ and fixed angular frequency ω propagating in a stratified shear flow, governed by the Taylor-Goldstein equation (1.10).

If we make the approximation that U and N^2 vary very slowly over a vertical wavelength, then we may use the ideas of ray theory and a local Galilean transformation to deduce from (2.3) that

$$\{\omega - U(z)/\alpha\}^2 = \alpha^2 N^2(z)/\{\alpha^2 + \gamma^2(z)\}\,. \qquad (30)$$

Sometimes $\omega - U(z)/\alpha$ is called the Doppler-shifted frequency. Formula (30) shows how γ varies as the wave propagates upwards. It can be seen that as the wave approaches a critical layer $\gamma^2 \to \infty$; thus the wave fronts become nearly horizontal and get closer together vertically (but not horizontally). The group velocity of the wave relative to the basic velocity of the fluid is zero and so the energy density E tends to infinity at the critical level. In fact, (Bretherton & Garrett 1968) the wave action $E/(\omega - U/\alpha)$ is a conserved quantity, i.e. independent of height in this approximation of ray theory, not E itself.

References

Banks, W.H.H., Drazin, P.G. and Zaturska, M.B., 1976 On the normal modes of parallel flow of inviscid stratified fluid. J. Fluid Mech. 58, 149-171.

Booker, J.R. and Bretherton, F.P., 1967 The critical layer for internal gravity waves in a shear flow. J. Fluid Mech. 27, 513-539.

Bretherton, F.P. and Garrett, C.J.R., 1968 Wavetrains in inhomogeneous moving media. Proc. Roy. Soc. A 302, 529-554.

Chandrasekhar, S., 1961 Hydrodynamic and Hydromagnetic Stability. Oxford University Press.

Chimonas, G., 1979 Algebraic disturbances in stratified shear flows. J. Fluid Mech. 90, 1-19.

Drazin, P.G. and Su, C.-H., 1975 A note on long-wave theory of airflow over a mountain. J. Atmos. Sci. 32, 437-439.

Eliassen, A. and Palm, E., 1961 On the transfer of energy in mountain waves. Geophys. Publ. 22, No. 3.

Goldstein, S., 1931 On the stability of superposed streams of fluid of different densities. Proc. Roy. Soc. A 132, 524-548.

Gossard, E.E. and Hooke, W.H., 1975 Waves in the Atmosphere. Amsterdam: Elsevier.

Haurwitz, B., 1931 Zur Theorie der Wellen in Luft and Wasser Veröff. Geophys. Inst., Univ. Leipzig, 6, No. 1.

Hazel, P., 1972 Numerical studies of the stability of inviscid stratified shear flow. J. Fluid Mech. 51, 39-61.

Howard, L.N., 1961 Note on a paper of John W. Miles. J. Fluid Mech 10, 496-508.

Krauss, W., 1966 Methoden und Ergebnisse der Theoretischen Ozeanographie II. Interne Wellen. Berlin: Gebrueder Borntraeger.

Lighthill, M.J., 1978 Waves in Fluids. Cambridge University Press.

Miles, J.W., 1961 On the stability of heterogeneous shear flows. J. Fluid Mech. 10, 496-508.

Rayleigh, Lord, 1883 Investigations on the character of the equilibrium of an incompressible fluid of variable density. Proc. Lord Math. Soc. 14, 170-177.

Synge, J.L., 1933 The stability of heterogeneous fluids. Trans. Roy. Soc. Canada 27, III, 1-18.

Taylor, G.I., 1931 Effect of variation of density on the stability of superposed streams of fluid. Proc. Roy. Soc. A 132, 499-523.

Titchmarsh, E.C., 1948 Introduction to the Theory of Fourier Integrals. Oxford University Press.

Yih, C.-S., 1955 Stability of two-dimensional parallel shear flows for three-dimensional disturbances. Quart. Appl. Math. 12, 434-435.

Yih, C.-S., 1965 Dynamics of Nonhomogeneous Fluids. New York: Macmillan.