TECHNICAL MEMORANDUM

39

Higher order closure schemes

L. Dümenil

Research Department

September 1981

European Centre for Medium-Range Weather Forecasts
Europäisches Zentrum für mittelfristige Wettervorhersage
Centre européen pour les prévisions météorologiques à moyen

1. EQUATIONS FOR THE MEAN MOTIONS AND FOR THE FLUCTUATING COMPONENTS

Reynolds (1895) derived a set of equations for the mean properties of a turbulent atmospheric shear layer.

Physical variables are then expressed as the sum of a mean value of the quantity ($\bar{u}$) and an instantaneous fluctuation of the quantity about its mean(u').

$$u = \bar{u} + u' \tag{1}$$

$$\bar{u} = \frac{1}{\Delta x\ \Delta y\ \Delta z\ \Delta t} \int_{\Delta x} \int_{\Delta y} \int_{\Delta z} \int_{\Delta t} u\ dx\ dy\ dz\ dt \tag{2}$$

In most applications

$$\overline{u'} \equiv 0, \tag{3}$$

which is, however, only strictly valid for special integration intervals (see Bernhardt, 1980).

Reynolds averaging can be applied to the momentum equations, continuity equation and thermodynamic energy equation simplified by a Boussinesq approximation.

The Reynolds equations for the mean flow are

$$\frac{\partial \bar{u}_k}{\partial x_k} = 0 \tag{4}$$

$$\frac{\partial \bar{u}_j}{\partial t} + \bar{u}_k \frac{\partial \bar{u}_j}{\partial x_k} + \epsilon_{jkl} f_k \bar{u}_l + \frac{\partial}{\partial x_k} \overline{u'_k u'_j}$$

$$= -\frac{1}{\rho_0} \frac{\partial \bar{p}}{\partial x_j} + \beta g_j \bar{\theta} + \nu \frac{\partial^2 \bar{u}_j}{\partial x_k \partial x_k} \tag{5}$$

$$\frac{\partial \bar{\theta}}{\partial t} + \bar{u}_k \frac{\partial \bar{\theta}}{\partial x_k} + \frac{\partial}{\partial x_k} \overline{u'_k \theta'} = \alpha \frac{\partial^2 \bar{\theta}}{\partial x_k \partial x_k} \tag{6}$$

$\beta \equiv \frac{1}{\theta_o}$	thermal expansion coefficient
ν	kinematic viscosity
α	thermal diffusivity
$u_i = \bar{u}_i + u'_i$	velocity components
$p = \bar{p} + p'$	pressure
$\theta = \bar{\theta} + \theta'$	potential temperature
$g_j = (0,0,-g)$	gravity vector

where p and θ are deviations from the value of these quantities in an atmosphere at rest described by θ_o, p_o, ρ_o. In the following, ρ_o will be set equal to 1.

Subtracting the equations for the mean flow from the Boussinesq equations leads to the equations for the fluctuating components

$$\frac{\partial u'_k}{\partial x_k} = 0 \tag{7}$$

$$\frac{\partial u'_j}{\partial t} + u'_k \frac{\partial u'_j}{\partial x_k} + \bar{u}_k \frac{\partial u'_j}{\partial x_k} + u'_k \frac{\partial \bar{u}_j}{\partial x_k} - \frac{\partial}{\partial x_k} \overline{u'_j u'_k}$$

$$+ \epsilon_{jkl} f_k u'_l = -\frac{1}{\rho_o} \frac{\partial p'}{\partial x_j} + \beta g_j \theta' + \nu \frac{\partial^2 u'_j}{\partial x_k \partial x_k} \tag{8}$$

$$\frac{\partial \theta'}{\partial t} + u'_k \frac{\partial \theta'}{\partial x_k} + \bar{u}_k \frac{\partial \theta'}{\partial x_k} + u'_k \frac{\partial \bar{\theta}}{\partial x_k} - \frac{\partial}{\partial x_k} \overline{\theta' u'_k} = \alpha \frac{\partial^2 \theta'}{\partial x_k \partial x_k} \tag{9}$$

Eqns. (4) - (6) introduce new unknowns, the Reynolds stress and heat conduction moments $\overline{u'_i u'_j}$, $\overline{u'_i \theta'}$, respectively. Therefore, the system of equations is no longer closed.

It is easy to derive equations for these unknowns, the second order correlations, from Eqns. (7) - (9) (see Busch, 1973).

$$\frac{\partial}{\partial t}\overline{u_i'u_j'} + \bar{u}_k\frac{\partial}{\partial x_k}\overline{u_i'u_j'} = -\frac{\partial}{\partial x_k}\overline{u_k'u_i'u_j'} \tag{10}$$

changes along a streamline

'velocity diffusion term', diffusion of $\overline{u_i'u_j'}$ by velocity fluctuations, divergence of vertical stress fluxes

$$+\nu\frac{\partial^2}{\partial x_k^2}\overline{u_i'u_j'} \qquad -\frac{\partial}{\partial x_j}\overline{p'u_i'} - \frac{\partial}{\partial x_i}\overline{p'u_j'}$$

diffusion of $\overline{u_i'u_j'}$ by molecular viscosity

pressure diffusion terms

$$-\overline{u_k'u_i'}\frac{\partial\bar{u}_j}{\partial x_k} - \overline{u_k'u_j'}\frac{\partial\bar{u}_i}{\partial x_k}$$

production or destruction of correlations by interaction of the fluctuating and the mean flow

$$+\beta\left(g_j\overline{u_i'\theta'} + g_i\overline{u_j'\theta'}\right)$$

creation or destruction of Reynolds stresses due to buoyancy forces

$$+\overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} \qquad -2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}$$

'pressure scrambling term' or 'tendency-toward-isotropy term', rearrangement of turbulent energy among velocity components

viscous dissipation

$$-f_k\left(\epsilon_{jkl}\overline{u_l'u_i'} + \epsilon_{ikl}\overline{u_l'u_j'}\right)$$

Coriolis terms

There are similar equations for the $\overline{u_i'\theta'}$ and $\overline{\theta'^2}$ (see Mellor 1973):

$$\frac{\partial}{\partial t}\overline{u_j'\theta'} + \bar{u}_k \frac{\partial}{\partial x_k}\overline{u_j'\theta'} + \frac{\partial}{\partial x_k}\overline{u_k' u_j' \theta'}$$

$$- \frac{\partial}{\partial x_k}\left[\alpha\, \overline{u_j' \frac{\partial \theta'}{\partial x_k}} + \nu\, \overline{\theta' \frac{\partial u_j'}{\partial x_k}}\right] + \frac{\partial}{\partial x_j}\overline{p'\theta'}$$

$$+ \epsilon_{jkl} f_k \overline{u_l'\theta'} = -\overline{u_j' u_k'}\frac{\partial \bar{\Theta}}{\partial x_k} - \overline{\theta' u_k'}\frac{\partial \bar{u}_j}{\partial x_k}$$

$$+ \beta g_j \overline{\theta'^2} + \overline{p'\frac{\partial \theta'}{\partial x_j}} - (\alpha + \nu)\overline{\frac{\partial u_j'}{\partial x_k}\frac{\partial \theta'}{\partial x_k}} \qquad (11)$$

$$\frac{\partial}{\partial t}\overline{\theta'^2} + \bar{u}_k \frac{\partial \overline{\theta'^2}}{\partial x_k} + \frac{\partial}{\partial x_k}\overline{u_k' \theta'^2} - \alpha \frac{\partial^2 \overline{\theta'^2}}{\partial x_k \partial x_k}$$

$$= -2\,\overline{u_k'\theta'}\frac{\partial \bar{\Theta}}{\partial x_k} - 2\alpha\,\overline{\frac{\partial \theta'}{\partial x_k}\frac{\partial \theta'}{\partial x_k}} \qquad (12)$$

The first four terms in Eqn. (10) show how diffusion acts on Reynolds stresses. There are contributions of velocity fluctuations, molecular viscosity and pressure fluctuations, the last one being least understood because of the difficulties in measuring the quantities involved.

The next two terms represent the production/destruction of stresses due to interaction between the eddy and the mean flow. The energy budgets for the mean flow and the turbulent flow can be obtained by contracting Eqn. (10) and from Eqn. (5). As the production term occurs in both equations with opposite signs, the mean flow supplies the energy for the turbulent motions.

The following term describes how buoyancy terms produce or destroy Reynolds stresses.

The next term is called the tendency-towards-isotropy-term which will be explained in more detail later. This term appears in the equations for $\overline{u_1'^2}$, $\overline{u_2'^2}$, $\overline{u_3'^2}$, but drops out of the equation for the total energy $\overline{u_1'^2} + \overline{u_2'^2} + \overline{u_3'^2}$ by virtue of the vanishing divergence of the turbulent velocity field. It therefore represents a rearrangement of the turbulent energy among the various components of velocity.

The last two terms represent the viscous dissipation and the effects of Coriolis accelerations, respectively. Viscous dissipation transforms kinetic energy of the turbulent motions $\overline{u_i'^2}$ into internal energy.

2. THE CLOSURE PROBLEM

Although these equations are an important tool to describe turbulent flows, they do not allow a direct solution of the problem, the prediction of second order correlations. Since the equations are non-linear, the additional equations for second order moments will require information on third order correlations and so on to correlations of order n involving those of order n+1. Therefore, there will always be one more unknown correlation than equations.

Several authors have derived methods to close the system of equations. The simplest reasonable one is the eddy viscosity or eddy diffusivity approach. Molecular viscosity in laminar flows enables molecular motions to transfer momentum and heat. In a turbulent flow, velocity fluctuations transfer these quantities. Molecular viscosity is then replaced by eddy viscosity defined as

$$\overline{u_i' u_j'} = - K \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \tag{13}$$

(K is also called turbulent exchange coefficient). K-theory and also the mixing length theory proposed by Prandtl are called first order closure schemes. They can only successfully describe a limited range of flows.

Mellor (1973), Wyngaard and Coté (1974), Deardorff (1972) and Sommeria (1976) have used higher order closure schemes. Correlations of a certain order are then related to the mean values and/or the correlations of lower order.

The above-mentioned authors have parameterized triple correlations in order to obtain estimates of second order moments. (These schemes are mostly called second order closure schemes. Some authors, however, call them third order closure schemes, because the third order correlations are parameterized). André et al (1976) provide model assumptions for fourth order correlations.

Higher order closure schemes are an alternative method for calculating the structure of the turbulent flows.

Only ensemble average statistics of the flow are considered. This reduces the computational effort which would be necessary to provide a three-dimensional simulation of the flow, where instantaneous details of the randomly fluctuating fields would be calculated. A three-dimensional simulation of a diurnal cycle (Deardorff, 1974) e.g. took roughly 350 hr on a CDC 7600 computer.

3. MODELLING OF CERTAIN TERMS

Modelling of second order correlations requires a lot of assumptions. Lumley and Khajeh-Nouri (1974) show that they can be minimized and that the model constants involved can be established through calculations of simpler flows.

In the following, the terms which are modelled in a second order closure scheme are listed. Details on the method of modelling are only given for the terms in the equations for the Reynolds stresses. A similar approach is possible for the $\overline{u_i'\theta'}$.

Rotta (1951) was the first to give a set of model assumptions for a certain term. The assumptions have been tested and the constants involved have been adjusted. Today, these models are widely used and referred to as a 'standard set of assumptions' (described in Deardorff (1973) and Donaldson (1973)).

If some means can be found to model the velocity diffusion terms, the terms containing pressure fluctuations (tendency-toward-isotropy and pressure diffusion terms) and the dissipation in terms of the mean variables and/or in terms of the second order correlations, then the set of equations is closed and a solution can be found.

When selecting a model for this purpose, the following basic principles (see Donaldson, 1973) must be observed:

1) The model term must be of tensor form so that it is invariant under an arbitrary transformation of coordinate systems. It must thus have all the tensor properties and, in addition, all the symmetries of the term which it replaces.

2) The model term must have the dimensional properties of the term it replaces.

3) The model term must satisfy all conservation relationships.

I. Modelling of the velocity diffusion terms $\overline{u_i'u_j'u_k'}$

The simplest tensor of rank three that can be obtained from the second-order correlations $\overline{u_i'u_k'}$ that is of the form T_{ijk} is

$$\frac{\partial}{\partial x_j} \overline{u_i' u_k'} .$$

The tensor $\overline{u_i'u_j'u_k'}$ is symmetric in all three indices so that the model must be symmetric in all these indices. Mellor (1973) chooses:

$$\overline{u_i' u_j' u_k'} \approx \frac{\partial}{\partial x_i} \overline{u_j' u_k'} + \frac{\partial}{\partial x_j} \overline{u_i' u_k'} + \frac{\partial}{\partial x_k} \overline{u_i' u_j'} \qquad (14)$$

To obtain correct dimensions, the model expression must be multiplied by a scalar velocity which is chosen as

$$q = \left(\overline{u_m' u_m'}\right)^{1/2} \qquad (15)$$

and by a scalar length λ_1, which is to be related to the scale of the mean motion or to the scale of the turbulence.

The minus sign in

$$\overline{u_i' u_j' u_k'} = -\lambda_1 q \left[\frac{\partial}{\partial x_k} \overline{u_i' u_j'} + \frac{\partial}{\partial x_j} \overline{u_i' u_k'} + \frac{\partial}{\partial x_i} \overline{u_j' u_k'} \right] \tag{16}$$

ensures that energy will be diffused down the gradient.

A similar approach can be made for $\overline{u'_k \theta'^2}$

$$\overline{u_k' \theta'^2} = -\lambda_2 q \frac{\partial \overline{\theta'^2}}{\partial x_k} \tag{17}$$

and for

$$\overline{u_k' u_j' \theta'} = -\lambda_3 q \left(\frac{\partial}{\partial x_j} \overline{u_k' \theta'} + \frac{\partial}{\partial x_k} \overline{u_j' \theta'} \right) \tag{18}$$

II. Modelling of the tendency-toward-isotropy term $\overline{p' \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right)}$

This is a second order tensor of the form T_{ij}. The simplest tensor of this form that can be formed from the second order correlations is the correlation $\overline{u_i' u_j'}$ itself, which has the right symmetry properties, but not the right dimensions.

If the term is modelled by

$$\frac{q}{\ell_1} \overline{u_i' u_j'} ,$$

there is a correction to be made: if i is set equal to j, the term to be modelled vanishes in an incompressible fluid while $q/\ell_1 \; \overline{u_i'u_i'}$ does not.

A reasonable correction is therefore

$$\overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} = -\frac{q}{3\ell_1}\left(\overline{u_i'u_j'} - \frac{\delta_{ij}}{3} q^2\right) \tag{19}$$

Eqn. (19) states that the tendency-toward-isotropy of the term $\overline{u_i'u_j'}$ is proportional to the degree of anisotropy.

The minus sign in this expression is chosen so that in the absence of other influences the turbulent energy will be equally distributed between the various components of velocity. Expression (19) was first given by Rotta (1951). It only includes the effect of mechanical turbulence. The pressure fluctuations p', however, depend (as can be seen from the Poisson equation for p' which follows from (8), see e.g. Wyngaard, Coté and Rao, 1974) on three methanisms:

(1) turbulence (the contribution of which is modelled by Eqn. (19))

(2) buoyancy and shear of the mean flow

(3) Coriolis forces.

The contributions of Coriolis forces on pressure correlations are negligibly small and therefore they are not included in the model.

A simplified model of the contributions from the mean shear and from buoyancy is proposed by Launder et al (1975):

$$C_1 q^2\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - C_2\left(P_{ij} - \frac{1}{3}\delta_{ij} P_{kk}\right),$$

where the P_{ij} include the total production/destruction of stresses by both shear and buoyancy (De Moor and André, 1975):

$$P_{ij} = -\overline{u_i'u_k'}\frac{\partial \bar{u}_j}{\partial x_i} - \overline{u_j'u_k'}\frac{\partial \bar{u}_i}{\partial x_j} + \beta g_j \overline{u_i'\theta'} + \beta g_i \overline{u_j'\theta'} \; .$$

In most cases, however, this production has been neglected as by Crow (1968).

Mellor (1973) takes Rotta's and Crow's terms into account, i.e.

$$\overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} = -\frac{q}{3\ell_1}\left(\overline{u_i' u_j'} - \frac{\delta_{ij}}{3} q^2\right) + C_1 q^2 \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{20}$$

III. Modelling of the dissipation term $\overline{2\nu \frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}}$

In the equations for $\overline{u_1'^2}$, $\overline{u_2'^2}$, $\overline{u_3'^2}$ this term describes the conversion of turbulent energy to heat by the action of molecular viscosity (Donaldson, 1973). It represents the rate at which viscous stresses perform deformation work against the fluctuating strain rate (Tennekes and Lumley, 1972). This is always a sink of kinetic energy. In the equations for the turbulent components of the flow the viscous dissipation cannot easily be neglected because it is essential to the dynamics of turbulence. In the equations for the mean flow, however, the corresponding term is negligibly small provided the Reynolds number is large.

Dissipation is connected with the smallest scales of turbulent flows which are generated by non-linear processes. At large Reynolds numbers the relative magnitude of viscosity v (1/R) is so small that viscous effects in a flow tend to become vanishingly small. The non-linear terms then, however, generate motions at scales small enough to be affected by viscosity. The smallest scale of motion automatically adjusts itself to the value of viscosity.

The generation of infinitely small scales of motion is prevented by the viscous terms dissipating small scale energy into internal heat. Kolmogorov's 'universal equilibrium theory' of the small scale structure is described by Tennekes and Lumley (1972). It is assumed that small scale motions with small time scales are statistically independent of the relatively slow large-scale turbulence and of the mean flow. Consequently:

1. Then the small-scale motion should depend only on the rate at which it is supplied with energy by the large-scale motion and on the kinematic viscosity. This rate of energy supply should be equal to the rate of dissipation (Tennekes and Lumley, 1972). If the amount of kinetic energy of the large scale turbulence is scaled by u^2, the rate of transfer of energy by $\frac{u}{\ell} u^2$ (where ℓ denotes the length scale of the largest eddies), the rate of energy supply is of order $\frac{u^3}{\ell}$. Taylor (1935) notes that with

$$\epsilon \sim \frac{u^3}{\ell} \tag{21}$$

the total dissipation of kinetic energy can be estimated from the large-scale dynamics which do not involve viscosity.

2. At large Reynolds numbers there is only very little direct interaction between fluctuations and the mean flow. Therefore, the small-scale structure of turbulence tends to be independent of orientation effects introduced by the mean shear. As all averages relating to the small eddies do not change under rotations or reflections of the coordinate system, Kolmogorov introduces 'local isotropy' for small-scale motions.

Assuming isotropy at large Reynolds numbers, Rotta (1951) notes that there are no correlations between the $\partial u'_i / \partial x_k$ and the $\partial u'_j / \partial x_k$ if $i \neq j$, so that

$$\nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}} = 0 \qquad \text{for } i \neq j. \tag{22}$$

Only the terms $\nu \overline{\left(\frac{\partial u'_i}{\partial x_k}\right)^2}$ give a contribution to the dissipation term

$$2\nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}}.$$

Under isotropic conditions, the contribution to the total dissipation of kinetic energy should be equal for all components i = 1,2,3. Then the dissipation term can be modelled as

$$2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} = \frac{2}{3} \frac{q^3}{\Lambda_1} \delta_{ij} . \tag{23}$$

IV. Other terms

The diffusion of $\overline{u_i' u_j'}$ by molecular viscosity

$$\nu \frac{\partial^2}{\partial x_k^2} \overline{u_i' u_j'}$$

is negligibly small at high Reynolds numbers.

Hanjalic and Launder (1972) state that the pressure diffusion terms are small in the first place. Mellor (1973) sets

$$\overline{p' u_i'} = \overline{p' \theta'} = 0 \tag{24}$$

4. CLOSED SET OF MODEL EQUATIONS

Simplifications and modelling assumptions lead to the following set of equations (Mellor, 1973)

$$\begin{aligned}
\frac{D}{Dt} \overline{u_i u_j} - \frac{\partial}{\partial x_k}\left[q \lambda_1 \left(\frac{\partial \overline{u_i' u_j'}}{\partial x_k} + \frac{\partial \overline{u_i' u_k'}}{\partial x_j} + \frac{\partial \overline{u_j' u_k'}}{\partial x_i} \right)\right] \\
= - \overline{u_k' u_i'} \frac{\partial \bar{u}_j}{\partial x_k} - \overline{u_k' u_j'} \frac{\partial \bar{u}_i}{\partial x_k} - \frac{2}{3} \frac{q^3}{\Lambda_1} \delta_{ij} \\
- \frac{q}{3 \ell_1} \left(\overline{u_i' u_j'} - \frac{\delta_{ij}}{3} q^2 \right) + c_1 q^2 \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \\
+ \beta \left(g_j \overline{u_i' \theta'} + g_i \overline{u_j' \theta'} \right) \\
+ f_k \left(\epsilon_{jkl} \overline{u_l' u_i'} + \epsilon_{ikl} \overline{u_l' u_j'} \right)
\end{aligned} \tag{25}$$

$$\frac{D}{Dt}\overline{\theta'^2} - \frac{\partial}{\partial x_k}\left[q\lambda_2 \frac{\partial \overline{\theta'^2}}{\partial x_k} \right] = -2\,\overline{u'_k\theta'}\,\frac{\partial \bar{\Theta}}{\partial x_k} - 2\,\frac{q}{\Lambda_2}\,\overline{\theta'^2} \tag{26}$$

$$\frac{D}{Dt}\overline{u'_j\theta'} - \frac{\partial}{\partial x_k}\left[q\lambda_3 \left(\frac{\partial \overline{u'_j\theta'}}{\partial x_k} + \frac{\partial \overline{u'_k\theta'}}{\partial x_j} \right)\right] \tag{27}$$

$$= -\overline{u'_j u'_k}\,\frac{\partial \bar{\Theta}}{\partial x_k} - \overline{u'_k\theta'}\,\frac{\partial \bar{u}_j}{\partial x_k} + \beta g_j\,\overline{\theta'^2} - \frac{q}{3\ell_2}\,\overline{u'_j\theta'}$$

All length scales are assumed proportional so that

$$\ell_1 = A_1\,\ell \qquad\qquad \ell_2 = A_2\,\ell$$

$$\Lambda_1 = B_1\,\ell \qquad\qquad \Lambda_2 = B_2\,\ell \tag{28}$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 0.23\,\ell$$

To obtain the length parameter ℓ, the mixing length profile as given by Blackadar (1962) can be used

$$\ell = \frac{k \cdot z}{1 + \frac{k \cdot z}{\ell_0}} \tag{29}$$

k von Kármán's constant

Miyakoda and Sirutis (1977) choose

$$l_o = 0.1 \frac{\int_0^h q z \, dz}{\int_0^h q \, dz} \qquad (30)$$

Yamada and Mellor (1979) introduce an additional prognostic equation for ℓ:

$$\begin{aligned} \frac{D}{Dt}(q^2\ell) - \frac{\partial}{\partial x_j}\left\{ q\ell \tilde{S}_\ell \frac{\partial}{\partial x_j}(q^2\ell)\right\} \\ = \ell E_1 \left(-\overline{u_j' u_i'} \frac{\partial \bar{u}_i}{\partial x_j} - \beta g_j \overline{u_j' \theta_v'} \right) \\ - \frac{q^3}{B_1} \left\{ 1 + E_2 \left(\frac{\ell}{k z}\right)^2 \right\} \end{aligned} \qquad (31)$$

The constants A_i, B_i, E_i, C_1 and the stability function $\tilde{S}_\ell$ can be derived empirically. Values are given by Mellor (1973) and Yamada and Mellor (1979).

5. SIMPLIFICATION OF EQUATIONS USING SCALING OF TERMS

The set of model equations containing 3 equations for the mean flow, Eqn. (4) - (6), and, in addition, 10 partial differential equations for the turbulent correlations, Eqn. (25) - (27), is far too complex and not economic. Mellor and Yamada (1974) (M-Y in the following) refer to this model as the Level 4 model. From this highest level of complexity, M-Y derived a hierarchy of 4 models, using a systematic way of simplification.

They use a parameter a which denotes the degree of anisotropy such that $a \to 0$ is the isotropic limit.

Although there exists no physical process such that $a \to 0$ as in the kinetic theory of gases (where $a \to 0$ as the mean free path approaches 0), M-Y assume that a is small enough for the purpose of simplifying the model.

After ordering the terms of the Level 4 model according to this parameter, terms of order a^2 and a, respectively, are neglected to decrease the complexity of the system.

Departures from isotropy are defined using a_{ij} and b_i as

$$\overline{u_i' u_j'} \equiv \left(\frac{\delta_{ij}}{3} + a_{ij} \right) q^2 \qquad , \quad a_{ii} = 0 \tag{32}$$

$$\overline{u_i' \theta'} \equiv b_i \varphi q \qquad , \quad \varphi = \overline{\theta'^2} , \tag{33}$$

where $\frac{1}{2} q^2 = \frac{1}{2} \overline{u_k'^2}$

The equations can be separated into equations for an isotropic and an anisotropic part, i.e. an equation for q^2, which follows from (25) by summation and an equation for a_{ij} q^2 which follows from (25) by using

$$\frac{\partial}{\partial t} \left(a_{ij} q^2 \right) = \frac{\partial}{\partial t} \overline{u_i' u_j'} - \frac{1}{3} \delta_{ij} \frac{\partial q^2}{\partial t} \tag{34}$$

Table 1 Scaling of terms

$$\frac{Dq^2}{Dt}-\frac{\partial}{\partial x_k}\left[q\lambda_1\frac{5}{3}\frac{\partial q^2}{\partial x_k}\{1+O(a)\}\right]=-2a_{ki}q^2\frac{\partial U_i}{\partial x_k}-2b_kg_k\beta q\varphi-2\frac{q^3}{\Lambda} \qquad (35,\mathrm{I})$$

Uq^2/L	Uq^2/L	aq^2U_x		$\beta bgq\varphi$	q^3/Λ
		q^3/Λ		q^3/Λ	q^3/Λ
$\frac{a^2q^3}{\Lambda}$	$\frac{a^2q^3}{\Lambda}$	$\frac{a^3q^3}{\Lambda}$	$\frac{q^3}{\Lambda}$	$\frac{q^3}{\Lambda}$	$\frac{q^3}{\Lambda}$

$$\frac{D}{Dt}(a_{ij}q^2)-\frac{\partial}{\partial x_k}\left[\frac{q\lambda_1}{3}\left\{\delta_{ik}\frac{\partial q^2}{\partial x_j}+\delta_{jk}\frac{\partial q^2}{\partial x_i}-\frac{2}{3}\delta_{ij}\frac{\partial q^2}{\partial x_k}\right\}\{1+O(a)\}\right]=-q^2\left[\left\{\frac{\delta_{ki}}{3}+a_{ki}\right\}\frac{\partial U_j}{\partial x_k}+\left\{\frac{\delta_{kj}}{3}+a_{kj}\right\}\frac{\partial U_i}{\partial x_k}\right.$$

aUq^2/L	Uq^2/L		$q^2U_x[1+O(a)]$	
			$a^{-1}q^3/\Lambda[1+O(a)]$	
$\frac{a^3q^3}{\Lambda}$	$\frac{a^2q^3}{\Lambda}$	$\frac{a^3q^3}{\Lambda}$	$\frac{1}{a}\frac{q^3}{\Lambda}$	$\frac{q^3}{\Lambda}$

$$\left.-\frac{2}{3}\delta_{ij}a_{kl}\frac{\partial U_l}{\partial x_k}-C_1\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\right]-\beta q\varphi(g_jb_i+g_ib_j-\tfrac{2}{3}\delta_{ij}g_lb_l)-\frac{q^3}{3l_1}a_{ij}$$

$b\beta q\varphi g$	aq^3/l
q^3/Λ	$a^{-1}q^3/\Lambda$
$\frac{q^3}{\Lambda}$	$\frac{1}{a}\frac{q^3}{\Lambda}$

(35,II)

$$\frac{D\overline{\theta^2}}{Dt}-\frac{\partial}{\partial x_k}\left[q\lambda_2\frac{\partial\overline{\theta^2}}{\partial x_k}\right]=-2q\varphi b_k\frac{\partial\Theta}{\partial x_k}-2\frac{q}{\Lambda_2}\overline{\theta^2} \qquad (35,\mathrm{III})$$

$U\varphi^2/L$	$U\varphi^2/L$	$q\varphi b\Theta_x$	$q\varphi^2/\Lambda$
		$q\varphi^2/\Lambda$	$q\varphi^2/\Lambda$
$\frac{b^2q}{\Lambda}$	$\frac{b^2q}{\Lambda}$	$\frac{q}{\Lambda}$	$\frac{q}{\Lambda}$

$$\frac{D}{Dt}(b_jq\varphi)-\frac{\partial}{\partial x_k}\left[q\lambda_3\left\{\frac{\partial}{\partial x_k}(b_jq\varphi)+\frac{\partial}{\partial x_j}(b_kq\varphi)\right\}\right]=-q^2\left(\frac{\delta_{jk}}{3}+a_{jk}\right)\frac{\partial\Theta}{\partial x_k}-q\varphi b_k\frac{\partial U_j}{\partial x_k}-g_j\beta\overline{\theta^2}-q^2\varphi b_j/3l_2 \qquad (35,\mathrm{IV})$$

$bUq\varphi/L$	$bUq\varphi/L$	$q^2\Theta_x[1+O(a)]$		$q\varphi bU_x$	$g\beta\varphi^2$	$q^2\varphi b/l$
		$b^{-1}q^2\varphi/\Lambda[1+O(a)]$		$q^2\varphi/\Lambda$	$b^{-1}q^2\varphi/\Lambda$	$b^{-1}q^2\varphi/\Lambda$
$\frac{b^3q^2}{\Lambda}$	$\frac{b^3q^2}{\Lambda}$	$\frac{1}{b}\frac{q^2}{\Lambda}$	$\frac{q^2}{\Lambda}$	$\frac{q^2}{\Lambda}$	$\frac{1}{b}\frac{q^2}{\Lambda}$	$\frac{1}{b}\frac{q^2}{\Lambda}$

U,Θ mean values

u,θ deviations from mean

The scaling of the terms in (35) is based on the following considerations:

$$\begin{aligned}
\ell &= \mathcal{O}(\ell_1) = \mathcal{O}(\ell_2) \\
\Lambda &= \mathcal{O}(\Lambda_1) = \mathcal{O}(\Lambda_2) \\
a^2 &= \mathcal{O}(a_{ij}^2)\,, \qquad b^2 = \mathcal{O}(b_i^2) \\
u_x^2 &= \mathcal{O}\left[\left(\frac{\partial \bar{u}_i}{\partial x_j}\right)^2\right], \quad \Theta_x^2 = \mathcal{O}\left[\left(\frac{\partial \bar{\Theta}}{\partial x_i}\right)^2\right]
\end{aligned} \tag{36}$$

From neutral experimental data the ratio for the length scales ℓ/Λ has been determined to $\ell/\Lambda \simeq 0.05 - 0.10$ (Mellor, 1973).

Although in most shear flows, production and dissipation do not balance locally, they generally are of the same order of magnitude (Tennekes and Lumley, 1972). If dissipation, which is proportional to q^3/Λ, and production are regarded as dominant in Eqn. I, then

$$a q^2 u_x = \frac{q^3}{\Lambda}\,. \tag{37}$$

For the same reason, the first and third term on the right hand side of Eqn. II lead to

$$q^2 u_x = a\,\frac{q^3}{\ell} \tag{38}$$

A combination of these two relations yields:

$$a^2 = \frac{\ell}{\Lambda} \quad \text{and} \quad u_x = \frac{1}{a}\,\frac{q}{\Lambda} \qquad (39),(40)$$

If similar considerations are made for Eqn. (35) III and IV, this leads to

$$q \varphi b \Theta_x = \varphi^2 \frac{q}{\Lambda}$$

$$q^2 \Theta_x = q^2 \varphi \frac{b}{\ell} \qquad (41),(42)$$

which gives

$$b^2 = \frac{\ell}{\Lambda}$$

$$\Theta_x = \frac{1}{b} \frac{\varphi}{\Lambda} \qquad (43),(44)$$

so that a = b.

Experimental laboratory boundary layer measurements and boundary layer models show that

$$\frac{\lambda}{\Lambda} = \mathcal{O}(a^2) \qquad (45)$$

and

$$\frac{\partial}{\partial t}(\quad) = \mathcal{O}(\Lambda^{-1}) \qquad (46)$$

Therefore, the diffusion terms are considered to be of order

$$a^2 \frac{q^3}{\Lambda} .$$

The determination of a final form of the equations depends on a comparison of the relative magnitudes of the different terms. Therefore some a priori information is needed about the time scale associated with the advection of the variables.

M-Y use $D/Dt \sim u/L$ (where L remains undefined) for the scaling of tendency and advection terms (see Eqn. (35)).

According to M-Y, experience shows that advection and tendency terms are generally small. They use the same scaling for diffusion and advection: $\mathcal{O}\left(\frac{uq^2}{L}\right)$. This is based on the belief that diffusion will assume a length scale which will tend to bring the processes of advection and diffusion into balance (Schemm and Lipps, 1976).

Velocity and "length" scales of the advective terms (U,L) can then be related to the parameters q and Λ by using

$$\frac{u q^2}{L} = a^2 \frac{q^3}{\Lambda} \quad . \tag{47}$$

The time scale of the mean flow L/U is therefore of order $1/a^2$ larger than Λ/q .

This approach is different from the one used by Schemm and Lipps (1976).. They distinguish between resolvable motions and subgrid turbulence. In this particular case, homogeneous turbulence experiments have shown that there is only one true scale appropriate for both kinds of motion

$$\frac{u}{L} \sim \frac{q}{\Delta} \tag{48}$$

where Δ denotes the grid spacing and L the scaling parameter for tendency and advection terms.

6. HIERARCHY OF MODELS

In order to derive the level 3 model as a first level of simplification, all terms of order a^3 or smaller relative to terms O(1) are neglected in Eqn. (35) I to IV.

The simplified set of equations consists of three differential equations for the mean flow [Eqn. (4) - (6)], two differential equations for the turbulent quantities q^2 and $\overline{\theta'^2}$, and eight algebraic equations

Table 2 Corrected versions of model Levels 1, 2, 3.

LEVEL 3

$$\frac{Dq^2}{Dt}-\frac{\partial}{\partial x_k}\left[\frac{5}{3}q\lambda_1\frac{\partial q^2}{\partial x_k}\right]=-2\overline{u_k u_i}\frac{\partial U_i}{\partial x_k}-2\beta g_k\overline{u_k\theta}-2\frac{q^3}{\Lambda_1}$$

$$\overline{u_i u_j}=\frac{\delta_{ij}}{3}q^2-\frac{3l_1}{q}\Big[(\overline{u_k u_i}-C_1q^2\delta_{ki})\frac{\partial U_j}{\partial x_k}$$

$$+(\overline{u_k u_j}-C_1q^2\delta_{kj})\frac{\partial U_i}{\partial x_k}-\tfrac{2}{3}\delta_{ij}\overline{u_k u_l}\frac{\partial U_l}{\partial x_k}\Big]$$

$$-3\frac{l_1}{q}\beta(g_j\overline{u_i\theta}+g_i\overline{u_j\theta}-\tfrac{2}{3}\delta_{ij}g_l\overline{u_l\theta})$$

$$\frac{D\overline{\theta^2}}{Dt}-\frac{\partial}{\partial x_k}\left[q\lambda_2\frac{\partial\overline{\theta^2}}{\partial x_k}\right]=-2\overline{u_k\theta}\frac{\partial\Theta}{\partial x_k}-2\frac{q}{\Lambda_2}\overline{\theta^2}$$

$$\overline{u_j\theta}=-3\frac{l_2}{q}\left[\overline{u_j u_k}\frac{\partial\Theta}{\partial x_k}+\overline{\theta u_k}\frac{\partial U_j}{\partial x_k}+\beta g_j\overline{\theta^2}\right].$$

LEVEL 2

$$\frac{q^3}{\Lambda}=-\overline{u_k u_i}\frac{\partial U_i}{\partial x_k}-\beta g_k\overline{u_k\theta},$$

$$\overline{u_i u_j}=\frac{\delta_{ij}}{3}q^2-3\frac{l_1}{q}\Big[(\overline{u_k u_i}-C_1q^2\delta_{ki})\frac{\partial U_j}{\partial x_k}$$

$$+(\overline{u_k u_j}-C_1q^2\delta_{kj})\frac{\partial U_i}{\partial x_k}-\tfrac{2}{3}\delta_{ij}\overline{u_k u_l}\frac{\partial U_l}{\partial x_k}\Big]$$

$$-3\frac{l_1}{q}\beta[g_j\overline{u_i\theta}+g_i\overline{u_j\theta}-\tfrac{2}{3}\delta_{ij}g_l\overline{u_l\theta}],$$

$$\frac{q\overline{\theta^2}}{\Lambda_2}=-\overline{u_k\theta}\frac{\partial\Theta}{\partial x_k},$$

$$\overline{u_j\theta}=-3\frac{l_2}{q}\left[\overline{u_j u_k}\frac{\partial\Theta}{\partial x_k}+\overline{\theta u_k}\frac{\partial U_j}{\partial x_k}+\beta g_j\overline{\theta^2}\right].$$

LEVEL 1

$$\frac{q^3}{\Lambda}=-\overline{u_k u_i}\frac{\partial U_i}{\partial x_k}-\beta g_k\overline{u_k\theta},$$

$$\overline{u_i u_j}=\frac{\delta_{ij}}{3}q^2-ql_1\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right),$$

$$\overline{\theta^2}=-\frac{\Lambda_2}{q}\overline{u_k\theta}\frac{\partial\Theta}{\partial x_k},$$

$$\overline{u_j\theta}=-ql_2\frac{\partial\Theta}{\partial x_j}-\frac{3\beta l_2}{q}g_j\overline{\theta^2}.$$

U, Θ mean values

u, θ deviations from mean values

If all terms of order a^2 are also neglected, the Level 2 model results only containing algebraic equations.

To obtain the Level 1 model all terms except those of $O(1)$ are neglected. In the neutral case, the Level 1 and 2 models bear direct resemblance to eddy or mixing length models.

The Level 3 model with boundary layer approximations provides algebraic expressions to estimate the Reynolds stresses $\overline{w'u'}$, $\overline{w'v'}$ and $\overline{w'\theta'}$, which then can be used in the prognostic equations for the turbulent kinetic energy q^2 and for $\overline{\theta'^2}$. The equations for the Reynolds stresses are of the form

$$\overline{w'u'} = -K_M \frac{\partial \bar{u}}{\partial z}$$

$$\overline{w'v'} = -K_M \frac{\partial \bar{v}}{\partial z} \qquad (49,50,51)$$

$$\overline{w'\theta'} = -K_H \frac{\partial \bar{\theta}}{\partial z}$$

where K_M and K_H can be regarded as sophisticated expressions for the exchange coefficients for momentum and heat, respectively, which are defined as follows:

$$\begin{aligned} K_M = {} & l_1 \Big[(1-3c)q^5 + 3l_2 \Big\{ (\Lambda_2 - 3l_2) q^3 \\ & + 3(4l_1 + \Lambda_2) c q^3 \Big\} \beta g \frac{\partial \bar{\theta}}{\partial z} \Big] \\ & \div \Big[q^4 + 6l_1^2 q^2 \Big\{ \Big(\frac{\partial \bar{u}}{\partial z}\Big)^2 + \Big(\frac{\partial \bar{v}}{\partial z}\Big)^2 \Big\} \\ & - 3l_1 l_2 \beta g \frac{\partial \bar{\theta}}{\partial z} \Big\{ 6l_1 (\Lambda_1 - 3l_2) * \\ & \Big\{ \Big(\frac{\partial \bar{u}}{\partial z}\Big)^2 + \Big(\frac{\partial \bar{v}}{\partial z}\Big)^2 + (7 + \Lambda_2/l_1) q^2 \\ & - 9l_2 (4l_1 + \Lambda_2) \beta g \frac{\partial \bar{\theta}}{\partial z} \Big\} \Big] \end{aligned} \qquad (52)$$

where $c = 0.056$

$$\begin{aligned} K_H = {} & l_2 \Big[q^3 - 6l_1 K_M \Big\{ \Big(\frac{\partial \bar{u}}{\partial z}\Big)^2 + \Big(\frac{\partial \bar{v}}{\partial z}\Big)^2 \Big\} \Big] \\ & \div \Big[q^2 - 3l_2 (4l_1 + \Lambda_2) \beta g \frac{\partial \bar{\theta}}{\partial z} \Big] \end{aligned} \qquad (53)$$

M-Y tested the performance of model Levels 2,3,4 on an idealized diurnal cycle, with an imposed surface temperature. All model levels gave approximately similar results. The difference between the most sophisticated Level 4 (ten differential equations for turbulence moments) and Level 3 (two differential equations) were less than between Level 3 and 2 results.

The Level 3 model was chosen to simulate the Wangara experiment (Yamada and Mellor, 1975). Experience derived from this successful numerical simulation lead to the proposal of a 'model of compromise' which was more economic. It was formally obtained by omitting advection, tendency and diffusion terms in all the equations for the second order correlations of the Level 3 model except in the equation for the turbulent energy. The resulting model was referred to as Level 2.5. It only requires the solution of a differential equation for the turbulent energy q^2. The rest of second moment equations are reduced to algebra.

In another simulation of the Wangara experiment (Yamada, 1977), the Level 2.5 reproduced well the results of the Level 3 model.

Miyakoda and Sirutis (1977) incorporated the Level 2.5 model in the GFDL general circulation model. They obtained significant improvement over previously tested parameterizations.

Yamada and Mellor (1979) use a one-dimensional version of the Level 2.5 model together with cloud ensemble relations in order to simulate BOMEX data. The results are said to be encouraging.

Strangely enough, Mellor and Yamada did not neglect the diffusion term in Eqn. (32) II in the 'hierarchy paper' (M-Y, 1974), although they wanted to eliminate small terms. Lipps (1977) pointed out that this term was (in our notation) $O(a^3)$ compared to the largest terms of $O(1)$ on the right hand side of the equation and could be deleted in order to obtain the Level 3 model.

Deletion of this term makes the further calculations slightly less complicated. The results of computations which include the term, are, however, said not to be significantly altered by its deletion. M-Y use the incorrect version of the model to simulate the Wangara data (Yamada and Mellor, 1975).

Miyakoda and Sirutis (1977) give a description of the corrected version used in the GFDL GC model. The documentation of the model (1975) and a listing of the code used in the GFDL model (1980), however, still show the additional terms.

References

André, J.C., G. De Moor, P. Lacarrère and R. du Vachat 1976 Turbulence approximation for inhomogeneous flows. Part I: The clipping approximation. J.Atm.Sci., 33, 476-481. Part II: The numerical simulation of a penetrative convection experiment. J.Atm.Sci., 33, 482-491.

Bernhardt, K. 1980 Zur Frage der Gültigkeit der Reynoldsschen Postulate. Zeitschrift für Meteorologie, Bd. 30, Heft 6, 361-368.

Blackadar, A.K. 1962 The vertical distribution of wind and turbulent exchange in a neutral atmosphere. J.Geophys.Res., 67, 3095-3102.

Busch, N.E. 1973 On the mechanics of atmospheric turbulence. In: Workshop on Micrometeorology, AMS, 1-61.

Crow, S.C. 1968 Visco elastic properties of fine grained incompressible turbulence. J.Fluid Mech., 33, 1-20.

Deardorff, J.W. 1972 Numerical investigation of neutral and unstable planetary boundary layers. J.Atm.Sci., 29, 91-115.

\- 1973 Three-dimensional numerical modeling of the planetary boundary layer. In: Workshop on Micrometeorology, AMS, 271-311.

\- 1974a Three-dimensional numerical study of the height and mean structure of a heated planetary boundary layer. Boundary-Layer Meteor., 7, 87-106.

\- 1974b Three-dimensional numerical study of turbulence in an entraining mixed layer. Boundary-Layer Meteor., 7, 199-226.

De Moor, G. and J.C. André 1975 La turbulence dans la couche limite atmosphérique. Modelisation de la couche limite. La Météorologie VI, No. 3, 179-195.

Donaldson, C. du P. 1973 Construction of a dynamic model of the production of atmospheric turbulence and the dispersal of atmospheric pollutants. In: Workshop on Micrometeorology, AMS, 313-392.

Hanjalic, K. and B.E. Launder 1972 A Reynolds stress model of turbulence and its application to thin shear flows. J.Fluid Mech., Vol.52, 4, 609-638,

Launder, B.E., G.J. Reece and W. Rodi 1975 Progress in the development of a Reynolds stress turbulence closure. J.Fluid Mech., Vol. 68, 3, 537-566.

Lipps, F.B. 1977 Corrigenda, J.Atm.Sci., 34, 1482.

Lumley, J.L. and B. Khajeh-Nouri 1974 Computational modeling of turbulent transport. Advances in Geophysics, Vol. 18A, 169-192.

Mellor, G.L. 1973 Analytic prediction of the properties of stratified planetary surface layers. J.Atm.Sci., 30, 1061-1069.

Mellor, G.L. and T. Yamada 1974 A hierarchy of turbulence closure models for planetary boundary layers. J.Atm.Sci., 31, 1791-1806.

Miyakoda, K. and J. Sirutis 1977 Comparative integration of global models with various parameterized processes of subgrid-scale vertical transports: Description of the parameterizations. Contr. to Atm. Physics, 50, 445-487.

Reynolds, O. 1895 On the dynamic theory of incompressible viscous fluids and the determination of the criterion. Phil. Trans. Roy. Soc. London A186, 123-164.

Rotta, J. 1951 Statistische Theorie nichthomogener Turbulenz. Zeitschrift für Physik, Bd. 129, 577-572.

Schemm, C.E. and F.B. Lipps 1976 Some results from a simplified model of atmospheric turbulence. J.Atm.Sci., 33, 1021-1071.

Sommeria, G. 1976 Three-dimensional simulation of turbulent processes in an undisturbed trade wind boundary layer. J.Atm.Sci., 33, 216-241.

Taylor, G.I. 1935 The transport of vorticity and heat through fluids in turbulent motion. Proceedings Roy.Soc. London, A135, 685.

Tennekes, H. and J.L. Lumley 1972 A first course in turbulence. The MIT Press, Cambridge, Massachusetts.

Wyngaard, J.C. and O.R. Coté 1974 The evolution of a convective planetary boundary layer - a higher order closure model study. Boundary-Layer Meteor., 7, 289-308.

Wyngaard, J.C., O.R. Coté and K.S. Rao 1974 Modeling the atmospheric boundary layer. Advances in Geophysics, Vol. 18A, 193-211.

Yamada, T. 1977 A numerical simulation of pollutant dispersion in a horizontally homogeneous atmospheric boundary layer. Atmos. Environ, 1015-1024.

Yamada, T. and G.L. Mellor 1975 A simulation of the Wangara atmospheric boundary layer data. J.Atm.Sci., 32, 2309-2329.

Yamada, T. and G.L. Mellor 1979 A numerical simulation of BOMEX data using a turbulence closure model coupled with ensemble cloud relations. Quart.J.Roy.Met.Soc., 105, 915-944.