Application of semi-implicit and semi-Lagrangian integration schemes to a limited area atmospheric model.

André Robert, Tai Loy Yee and Harold C. Ritchie.

The formulation of a multi-level atmospheric model is discussed. In particular, the time integration scheme used in this model is described in detail. This scheme is both semi-implicit and semi-Lagrangian. The calculations are broken down into separate segments for each of three time levels. One of these segments involve interpolation of the variables to upstream points and another segment requires the solution of a Poisson equation in three dimensions. The description of the method is followed by an integration of the model with a time step of 90 minutes. The integration scheme appears to be stable and the forecast seems reasonable.

1. Introduction

A substantial number of atmospheric models use a semi-implicit algorithm in order to carry out the time integration. With such a scheme, one can use longer time steps and reduce the computer time required to produce predictions. In the semi-implicit scheme the size of the time step is no longer restricted by the phase speed of the external gravity wave. It is limited mainly by the magnitude of the wind associated with advection.

For problems that are strictly advective, it is sometimes desirable to use the Lagrangian technique. For such a problem, most Lagrangian schemes are unconditionally stable. In an atmospheric model, it might prove advantageous to combine a Lagrangian scheme with the semi-implicit algorithm.

Such an experiment was performed by Robert (1982). In this experiment, the shallow water equations were integrated with a time step of two hours. Another similar experiment was performed by Bates and McDonald (1982). Here, a multi-level model was used with a time step of one hour. In this model, a semi-Lagrangian scheme is combined with the split-explicit scheme. This version of their model (with Δt = 30 minutes) was incorporated into their forecast production system with a reduction of 38% in computer time.

Here also, a semi-Lagrangian scheme will be combined with the semi-implicit technique in a three dimensional forecast model. An integration will be performed in order to show that this scheme is stable. A detailed description of the proposed scheme will be given for those who may wish to apply it to their own model.

2. Formulation of the model

The meteorological equations will be given in a vertical sigma coordinate system where

$$\sigma = \frac{p}{p_s} \tag{1}$$

while p is pressure and p_s is the pressure at the ground (lower boundary).

These equations are

$$\frac{dU}{dt} - fV + \frac{\partial \phi}{\partial X} + RT\frac{\partial P}{\partial X} + K\frac{\partial S}{\partial X} = 0 \qquad (2)$$

$$\frac{dV}{dt} + fU + \frac{\partial \phi}{\partial Y} + RT\frac{\partial P}{\partial Y} + K\frac{\partial S}{\partial Y} = 0 \qquad (3)$$

$$\frac{dP}{dt} + S\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) + \frac{\partial \dot{\sigma}}{\partial \sigma} = 0 \qquad (4)$$

$$\frac{dT}{dt} - \frac{RT}{C_p}\left(\frac{dP}{dt} + \frac{\dot{\sigma}}{\sigma}\right) = 0 \qquad (5)$$

$$\frac{\partial \phi}{\partial \sigma} = -\frac{RT}{\sigma} \qquad \phi = gZ \qquad (6)$$

$$P = \ln p_s \qquad K = \tfrac{1}{2}(U^2+V^2) \qquad (7)$$

$$S = m^2 \qquad [U,V] = \left[\frac{u}{m}, \frac{v}{m}\right] \qquad (8)$$

In these equations, m is the map scale associated with a conformal projection, $[u, v]$ represents the true wind vector while $[U,V]$ is a suitable model wind vector with its components along the cartesian $[X,Y]$ coordinates of the projection.

As we can see from eqn(6), the hydrostatic approximation is used. These equations involve a few other minor approximations that will not be discussed here.

In order to apply the proposed algorithm to these equations, the vertical advection will be taken out of the total derivative and will be treated separately. Also, the temperature and the geopotential will be replaced by

$$T = T^* + T' \qquad (9)$$

$$\phi = G - RT^*P \qquad (10)$$

where T^* is a temperature that is only a function of σ. All terms containing both T^* and T' will be broken down in order to separate these variables. Finally, a time average will be applied to some of the terms. Our system of equations will then take the form

$$\frac{d_H U}{dt} + \dot{\sigma}\frac{\partial U}{\partial \sigma} - fV + \overline{\frac{\partial G}{\partial X}}^{t} + RT'\frac{\partial P}{\partial X} + K\frac{\partial S}{\partial X} = 0 \tag{11}$$

$$\frac{d_H V}{dt} + \dot{\sigma}\frac{\partial V}{\partial \sigma} + fU + \overline{\frac{\partial G}{\partial Y}}^{t} + RT'\frac{\partial P}{\partial Y} + K\frac{\partial S}{\partial Y} = 0 \tag{12}$$

$$\frac{d_H P}{dt} + \overline{S\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right)}^{t} + \overline{\frac{\partial \dot{\sigma}}{\partial \sigma}}^{t} = 0 \tag{13}$$

$$\frac{d_H T'}{dt} + \overline{\dot{\sigma}}^{t}\frac{\partial T^*}{\partial \sigma} + \dot{\sigma}\frac{\partial T'}{\partial \sigma} - \frac{RT^*}{C_p}\left(\frac{d_H P}{dt} + \overline{\frac{\dot{\sigma}}{\sigma}}^{t}\right) - \frac{RT}{C_p}\left[\frac{\dot{\sigma}}{\sigma} - \frac{\partial \dot{\sigma}}{\partial \sigma} - S\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right)\right] = 0 \tag{14}$$

At this point, we will start considering the calculations that will have to be carried out by the computer. These calculations will be broken down into segments and the first segment will consist in calculating all the explicit terms appearing in each of the four equations given above. The four corresponding results are obtained from:

$$\Omega_1 = \dot{\sigma}\frac{\partial U}{\partial \sigma} - fV + RT'\frac{\partial P}{\partial X} + K\frac{\partial S}{\partial X} \tag{15}$$

$$\Omega_2 = \dot{\sigma}\frac{\partial V}{\partial \sigma} + fU + RT'\frac{\partial P}{\partial Y} + K\frac{\partial S}{\partial Y} \tag{16}$$

$$\Omega_3 = 0 \tag{17}$$

$$\Omega_4 = \dot{\sigma}\frac{\partial T'}{\partial \sigma} - \frac{RT'}{C_p}\left[\frac{\dot{\sigma}}{\sigma} - \frac{\partial \dot{\sigma}}{\partial \sigma} - S\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right)\right] \tag{18}$$

and eqns(11) to (14) may now be reduced to

$$\frac{d_H U}{dt} + \overline{\frac{\partial G}{\partial X}}^{t} + \Omega_1 = 0 \tag{19}$$

$$\frac{d_H V}{dt} + \overline{\frac{\partial G}{\partial Y}}^{t} + \Omega_2 = 0 \tag{20}$$

$$\frac{d_H P}{dt} + \overline{S\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right)}^{t} + \overline{\frac{\partial \dot{\sigma}}{\partial \sigma}}^{t} = 0 \tag{21}$$

$$\frac{d_H T'}{dt} + \overline{\dot{\sigma}}^{t} \frac{\partial T^*}{\partial \sigma} - \frac{RT^*}{C_p}\left(\frac{d_H P}{dt} + \frac{\overline{\dot{\sigma}}^{t}}{\sigma}\right) + \mathcal{R}_4 = 0 \tag{22}$$

Spatial finite difference approximations are not considered in the formulation of the model given above. It will be assumed in the following sections that suitable finite differences are used in all three space dimensions.

3. The time integration procedure

In order to carry out the integration, values of the variables are required at three points called A^-, A^0 and A^+. The coordinates of these points are given as follows

$$A^+(X \quad , \quad Y \quad , t+\Delta t) \tag{23}$$

$$A^0(X-\alpha \, , \, Y-\beta \, , \, t \,) \tag{24}$$

$$A^-(X-2\alpha , Y-2\beta , t-\Delta t) \tag{25}$$

where X, Y represents a grid point and (α, β) represents a displacement. This displacement is computed from the wind at the mid point.

$$\alpha = \Delta t \, S^0 U^0 \tag{26}$$

$$\beta = \Delta t \, S^0 V^0 \tag{27}$$

Time derivatives and time averages will be computed from:

$$\frac{d_H F}{dt} = \frac{F^+ - F^-}{2\Delta t} \tag{28}$$

$$\overline{F}^{t} = \frac{F^+ + F^-}{2} \tag{29}$$

We will immediately introduce both approximations into eqns(19) to (22) and we will group together all terms evaluated at the same time level. At time $t+\Delta t$, we have the following terms

$$q_1 = U + \Delta t \frac{\partial G}{\partial X} \tag{30}$$

$$q_2 = V + \Delta t \frac{\partial G}{\partial Y} \tag{31}$$

$$q_3 = P + \Delta t \left[S\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) + \frac{\partial \dot{\sigma}}{\partial \sigma} \right] \tag{32}$$

$$q_4 = T' - \frac{RT^*}{C_p} P + \Delta t \left(\dot{\sigma} \frac{\partial T^*}{\partial \sigma} - \frac{RT^*}{C_p} \frac{\dot{\sigma}}{\sigma} \right) \tag{33}$$

On the other hand, at time $t - \Delta t$, we have

$$p_1 = U - \Delta t \frac{\partial G}{\partial X} \tag{34}$$

$$p_2 = V - \Delta t \frac{\partial G}{\partial Y} \tag{35}$$

$$p_3 = P - \Delta t \left[S\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) + \frac{\partial \dot{\sigma}}{\partial \sigma} \right] \tag{36}$$

$$p_4 = T' - \frac{RT^*}{C_p} P - \Delta t \left(\dot{\sigma} \frac{\partial T^*}{\partial \sigma} - \frac{RT^*}{C_p} \frac{\dot{\sigma}}{\sigma} \right) \tag{37}$$

Since the variables are known at time $t - \Delta t$, we can immediately compute p_1, p_2, p_3 and p_4. Our four differential equations will now take the form

$$\frac{q_K^+ - p_K^-}{2\Delta t} + \Omega_K^0 = 0 \qquad \text{FOR } K = 1, 2, 3, 4 \tag{38}$$

A quick review will be given at this point. The first segment of computations consists in using eqns(15 to (18) to evaluate Ω_K at grid points from the variables at time t. The second segment consists in using eqns(34) to (37) to evaluate p_K at grid points from the variables at time $t - \Delta t$. Now we can start the third segment of computations which consists in evaluating q_K

The first step consists in evaluating (α, β). For this purpose we will use an iterative procedure

$$\alpha^{n+1} = \Delta t \; SU \; (X-\alpha^n, Y-\beta^n, t) \tag{39}$$

$$\beta^{n+1} = \Delta t \; SV \; (X-\alpha^n, Y-\beta^n, t) \tag{40}$$

This procedure starts with the displacement (α, β) computed at the preceding time step and in general, it is found that two iterations are sufficient. Here we find that we have to interpolate in the horizontal to obtain values of SU and SV at the point given by the coordinates ($X-\alpha^n$, $Y-\beta^n$). A bicubic interpolation scheme is used for this purpose but one could use any other suitable interpolation formula. Several suitable algorithms are discussed by Huffenus and Khaletzky (1981) and also by Bates and McDonald (1982).

The next step consists in calculating $\mathcal{R}^o$ and p^- at the appropriate upstream points

$$\mathcal{R}^o = \mathcal{R}(X-\alpha, Y-\beta) \tag{41}$$

$$p^- = p(X-2\alpha, Y-2\beta) \tag{42}$$

and here also a bicubic algorithm will be used. The calculation proceeds from a set of four by four grid points selected in such a way that the specified upstream point lies inside the central square of this grid. This is the condition that guarantees the stability of the advective portion of the numerical integration.

The last step of this third segment of computations will consist in calculating .

$$q^+ = p^- - 2\Delta t \, \mathcal{R}^o \tag{43}$$

We are now ready to move on to the fourth segment of computations. It consists in solving the following set of equations

$$U + \Delta t \frac{\partial G}{\partial X} = q_1 \tag{44}$$

$$V + \Delta t \frac{\partial G}{\partial Y} = q_2 \tag{45}$$

$$P + \Delta t \left[S\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) + \frac{\partial \dot{\sigma}}{\partial \sigma} \right] = q_3 \tag{46}$$

$$T' - \frac{RT^*}{C_p} P + \Delta t \left(\dot{\sigma} \frac{\partial T^*}{\partial \sigma} - \frac{RT^*}{C_p} \frac{\dot{\sigma}}{\sigma} \right) = q_4 \tag{47}$$

for the variables U, V, $\dot{\sigma}$, T and P. This is the implicit portion of the computations and the solution is obtained by the method described previously by Robert, Henderson and Turnbull (1972). This method consists in using the elimination process to reduce eqns(44) to (47) to a single equation containing one variable. The result of this operation gives us

$$H = \frac{\partial}{\partial \sigma}\left(\frac{R q_4}{\sigma \gamma^*} + \sigma \tilde{q}_3\right) - \Delta t\left[\frac{q_3 - \tilde{q}_3}{\Delta t} - S\left(\frac{\partial q_1}{\partial X} + \frac{\partial q_2}{\partial Y}\right)\right] \tag{48}$$

$$\left(\frac{G_\sigma}{\gamma^*}\right)_\sigma + \Delta t^2 \nabla^2 G = H \tag{49}$$

As we can see from these expressions, we first compute H and then solve a three-dimensional Poisson equation for G. In these equations, we have

$$\gamma^* = \frac{R}{\sigma}\left(\frac{RT^*}{\sigma C_p} - \frac{\partial T^*}{\partial \sigma}\right) \tag{50}$$

$$\tilde{q}_3 = \int_0^1 q_3 \, \delta\sigma \tag{51}$$

Given G, we can compute the other variables

$$U = q_1 - \Delta t \frac{\partial G}{\partial X} \tag{52}$$

$$V = q_2 - \Delta t \frac{\partial G}{\partial Y} \tag{53}$$

$$P = \tilde{q}_3 - \Delta t \, S\left(\frac{\partial \tilde{U}}{\partial X} + \frac{\partial \tilde{V}}{\partial Y}\right) \tag{54}$$

$$\frac{\partial \dot{\sigma}}{\partial \sigma} = \frac{q_3 - P}{\Delta t} - S\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) \tag{55}$$

$$\phi = G - RT^* P \tag{56}$$

$$T = -\frac{\sigma}{R}\frac{\partial \phi}{\partial \sigma} \tag{57}$$

These computations complete a full cycle which is repeated until the desired forecast is obtained.

4. Numerical integration

The first run with this model was performed from the analysed variables at 1200 GMT, 3 DEC 1982. The surface pressure at that time is presented in FIG.1. For this particular run the model used a 33x33 grid with a mesh length of 254km at 60°N. Surface pressure is presented on a 13x13 window within the model grid. A 24 hour integration with a time step of 90 minutes was carried out and the resulting surface pressure is presented in FIG.2. The same integration was also performed with a step of 45 minutes and the result is given in FIG.3. Finally, the verifying analysis is given in FIG.4.

Another three similar integrations have been completed and from these four cases, one can conclude that there are no apparent signs of instability. The proposed scheme seems to be stable. Longer integrations have not been performed yet in an attempt to confirm this conclusion.

From the integrations shown in FIG.2 and FIG.3, we can see that the surface pressure is fairly sensitive to the size of the time step but we must not conclude from this that a time step of 90 minutes will give inaccurate predictions. A suitable initialization scheme has not yet been incorporated into the model and the mesh length of 254km is not adequate. A finer mesh will produce more accurate interpolations and reduce time truncation errors.

5. Conclusion

By combining a semi-Lagrangian scheme with the semi-implicit algorithm for the integration of the primitive meteorological equation, one can use substantially larger time steps. The resulting integrations appear to be computationally stable. It is not yet possible to say whether or not the integrations are accurate and efficient. The development of the model has to be completed before undertaking any experiments along these lines. Such tests are in preparation and results will be published as soon as they become available.

References

Bates, J.R. and A. McDonald, 1982: Multiply-upstream, semi-Lagrangian advective schemes: analysis and application to a multi-level primitive equations model. Mon. Wea. Rev., Vo. 110, pp. 1831-1842.

Huffenus, J.P. and D. Khaletzky, 1981: The Lagrangian approach of advective term treatment and its application to the solution of Navier-Stokes equations. Numerical Methods in Fluids, Vol. 1, pp. 365-388.

Robert, A., 1982: A semi-Lagrangian and semi-implicit numerical integration scheme for the primitive meteorological equations. J. Met. Soc. Japan, Vol. 60, pp. 319-325.

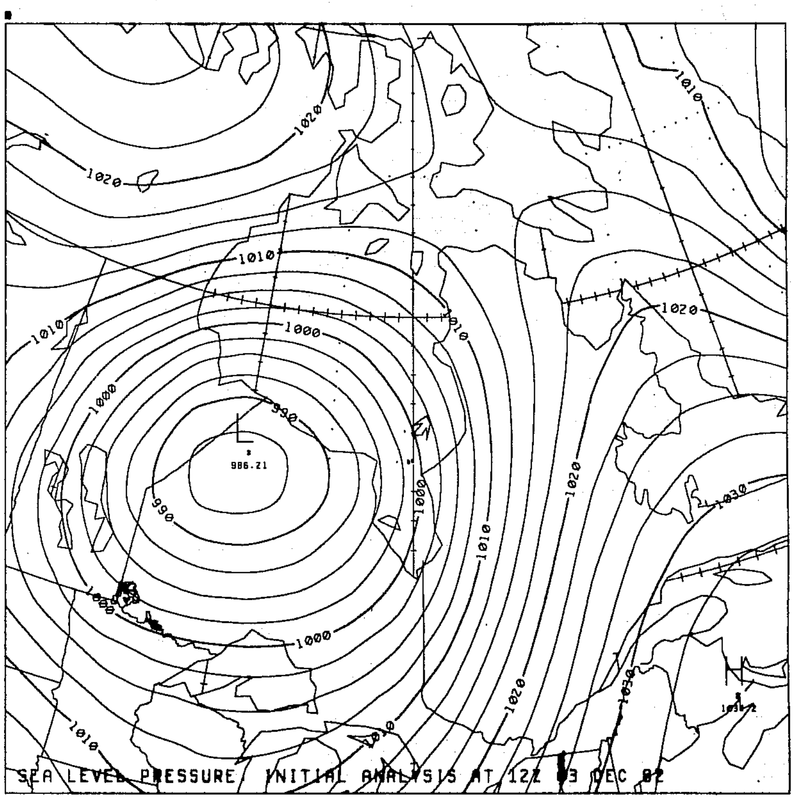

FIG. 1 Sea-level pressure analysis for 12:00 GMT 03 December 1982.

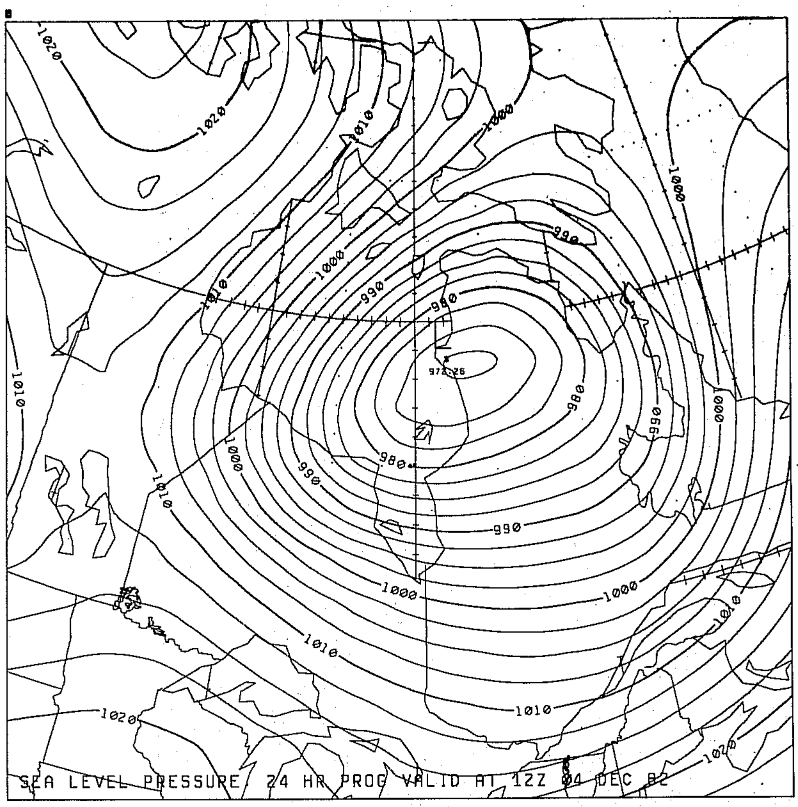

FIG. 2 24-hour forecast of sea-level pressure valid at 12:00 GMT 04 December 1982. Semi-Lagrangian and semi-implicit model with a timestep of 90 minutes.

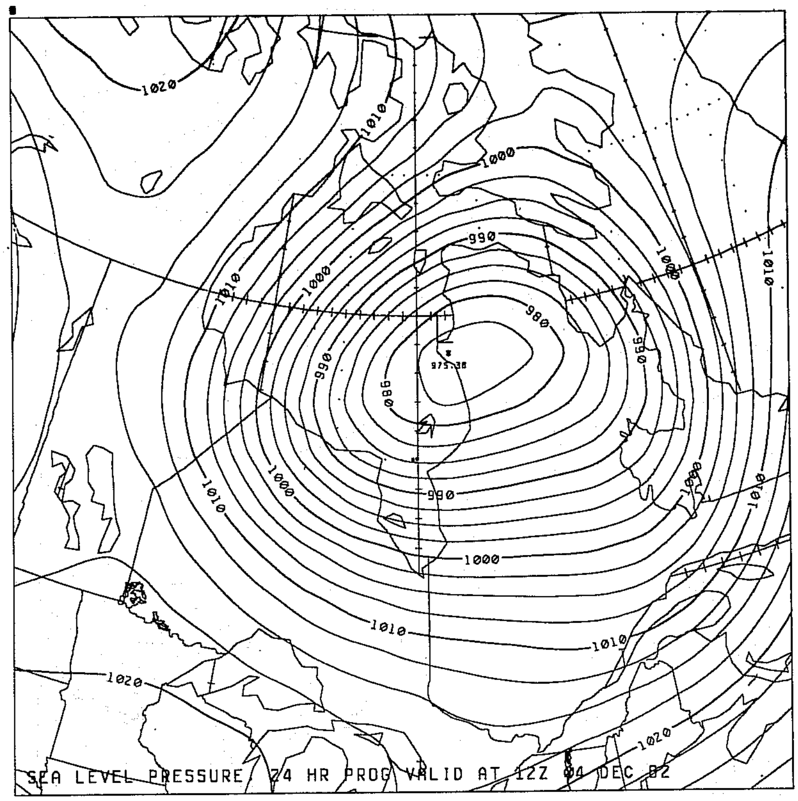

FIG. 3 24-hour forecast of sea-level pressure valid at 12:00 GMT 04 December 1982. Semi-Lagrangian and semi-implicit model with a time step of 45 minutes.

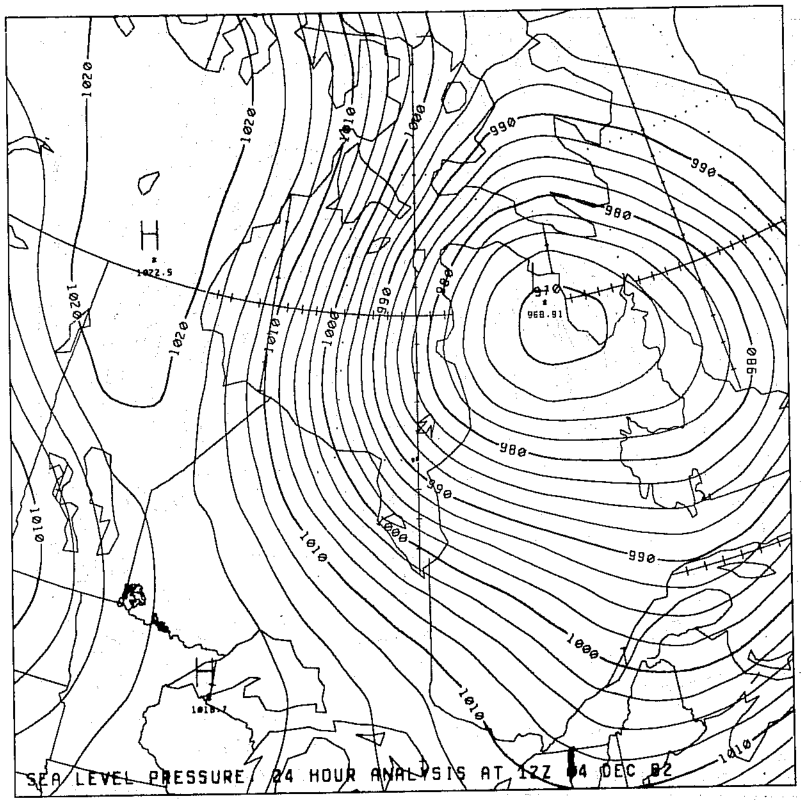

FIG. 4 Sea-level pressure analysis for 12:00 GMT 04 December 1982.