The design of efficient time integration schemes for the primitive equations.

André Robert

The step used in numerical integrations is frequently limited to unreasonably small values imposed by the requirement for stability. In some cases, the condition for stability can be relaxed through slight changes in the integration procedure. The semi-implicit technique and the Lagrangian method may be used for this purpose. Schemes combining both techniques will be examined and analysed for their stability properties. Results of an application to the shallow water equations will be presented.

1. Introduction

The explicit leapfrog time integration scheme is very inefficient when it is used in large scale atmospheric models. For a grid length of 200 km, a time step of the order of 5 minutes has to be used. It may even be necessary to reduce that time step to 2 minutes if a staggered grid and fourth order differences are used. Most explicit time integration schemes have to use rather short time steps.

It seems possible to design schemes that are unconditionnally stable and with such a scheme we might be able to obtain accurate results with a time step of 2 hours. If this is true, and if the required additional computations are not too time consuming, we might end up with a very efficient model. The computer time required to produce a forecast might be reduced by a factor of 10 or more.

In the following sections, we will propose an algorithm that combines the semi-implicit scheme, a semi-Lagrangian scheme and the leapfrog scheme. The stability of the proposed algorithm will be analysed for a linearized version of the shallow water equations.

An integration with the nonlinear shallow water equations will also be performed in order to show that the algorithm gives reasonably accurate results.

2. The proposed algorithm

In the shallow water equations, it is proposed to apply a time average to the Coriolis term, to the divergence and to the term involving the gradient of the geopotential. These time averages, as well as the total time derivatives will be evaluated by using points along the trajectories of the air parcels. The corresponding linearized equations take the form:

$$\frac{du}{dt} - f\overline{v}^t + \frac{\partial \overline{\phi}^t}{\partial x} = 0 \tag{1}$$

$$\frac{dv}{dt} + f\overline{u}^t + \frac{\partial \overline{\phi}^t}{\partial y} = 0 \tag{2}$$

$$\frac{d\phi}{dt} + v\frac{\partial \phi_0}{\partial y} + \phi_0\left(\frac{\partial \overline{u}^t}{\partial x} + \frac{\partial \overline{v}^t}{\partial y}\right) = 0 \tag{3}$$

where:

u is the deviation from the basic flow U
v is the deviation from the basic flow $V = 0$
ϕ is the deviation from ϕ_0

u, v and ϕ are assumed to be infinitesimally small

$$f U = -\frac{\partial \phi_0}{\partial y} \tag{4}$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + U \frac{\partial F}{\partial x} \tag{5}$$

where F is either u, v or ϕ. Both $f = 2\Omega$ and U are assumed to be constants.

In the absence of perturbations, if an underlying surface is introduced and selected in such a way that it is everywhere parallel to the free upper surface ϕ_0, then the term $v \frac{\partial \phi_0}{\partial y}$ in eqn(3) will be eliminated and the term ϕ_0 in front of the divergence must be replaced by the thickness of the fluid which is a constant in this case. In other words, we will replace eqn(3) by:

$$\frac{d\phi}{dt} + \phi_0 \left(\frac{\partial \bar{u}^t}{\partial x} + \frac{\partial \bar{v}^t}{\partial y} \right) = 0 \tag{6}$$

where ϕ_0 is a constant.

In order to carry out the calculations, three points P_1, P_2 and P_3 are defined with the following coordinates

$$P_1 (x , y , t+\Delta t) \tag{7}$$

$$P_2 (x-a, y-b, t) \tag{8}$$

$$P_3 (x-2a, y-2b, t-\Delta t) \tag{9}$$

Here we can see that P_2 is the mid point of the interval from P_1 to P_3. The displacements a and b will be calculated from the winds at P_2.

$$a = \Delta t [U(P_2) + u(P_2)] \tag{10}$$

$$b = \Delta t [V(P_2) + v(P_2)] \tag{11}$$

For infinitesimal perturbations, these displacements reduce to:

$$a = \Delta t \, U \tag{12}$$

$$b = 0 \tag{13}$$

The time derivatives and time averages will be evaluated as follows

$$\frac{dF}{dt} = \frac{F(P_1) - F(P_3)}{2\Delta t} \tag{14}$$

$$\overline{F}^{t} = \frac{F(P_1) + F(P_3)}{2} \tag{15}$$

In other words, we are taking a trajectory over a time interval of $2\Delta t$. This trajectory terminates at a grid point. The end points of this trajectory are used to compute the time derivatives and the time averages. Any explicit term appearing in the equations would be evaluated at the mid point of the trajectory. This means that the model uses centered differences of second order accuracy.

3. Stability analysis

An attempt will be made to find exact solutions of eqs(1),(2) and (6) in terms of the exponential

$$E = e^{i(\omega t + kx + ly)} \tag{16}$$

For this function we have

$$\frac{dE}{dt} = \frac{1}{2\Delta t}\left[e^{i\omega\Delta t} - e^{-i(\omega\Delta t + 2ka)} \right] E \tag{17}$$

$$\frac{dE}{dt} = \frac{1}{2\Delta t}\left[e^{i(\omega\Delta t + ka)} - e^{-i(\omega\Delta t + ka)} \right] E(P_2) \tag{18}$$

$$\frac{dE}{dt} = \frac{i}{\Delta t} E(P_2) \sin(\omega + kU)\Delta t \tag{19}$$

and in a similar fashion we also have

$$\overline{E}^{t} = E(P_2) \cos(\omega + kU)\Delta t \tag{20}$$

Substitution in the shallow water equations gives the following results

$$\frac{i}{\Delta t} S u_2 - f C v_2 + i k C \phi_2 = 0 \quad (21)$$

$$\frac{i}{\Delta t} S v_2 + f C u_2 + i l C \phi_2 = 0 \quad (22)$$

$$\frac{i}{\Delta t} S \phi_2 + i \phi_0 C (k u_2 + l v_2) = 0 \quad (23)$$

where

$$S = \sin(\omega + kU)\Delta t \quad (24)$$

$$C = \cos(\omega + kU)\Delta t \quad (25)$$

We have an homogeneous system of equations in terms of the variables u, v and ϕ. For this system to have non trivial solutions, its determinant must vanish. This gives the frequency equation

$$S \left\{ S^2 - C^2 \Delta t^2 [f^2 + \phi_0 (k^2 + l^2)] \right\} = 0 \quad (26)$$

and from this equation we obtain the following frequencies

$$\omega = -kU$$

$$\omega = -kU \pm \frac{1}{\Delta t} \tan^{-1} \Delta t [f^2 + \phi_0 (k^2 + l^2)]^{1/2} \quad (27)$$

It is quite clear from this result that the solutions are unconditionally stable.

4. Numerical integrations

The shallow water equations will be integrated on a 61 by 61 grid in a polar stereographic projection with a grid length of 190.5 km at 60^{o}N. Fourth order differences are used to compute all space derivatives and fourth order interpolation is used to compute upstream values along the trajectories.

The numerical integration is performed from the 500mb analysed variables at 12:00 GMT 30 August 1981. The geopotential is presented in FIG.1 followed by a 48 hour prediction shown in FIG.2. This prediction was produced by the semi-Lagrangian and semi-implicit model of the shallow water equations with a time step of two hours. The result obtained with a time step of one hour is also shown in FIG.3. The rather small differences between these integrations indicate that the truncation errors associated with this integration scheme are small even with a time step as large as two hours.

5. Conclusion

A semi-Lagrangian time integration scheme can easily be combined with the semi-implicit algorithm in a model of the shallow water equations. This combination enables us to increase the time step by another factor of four to six over the strictly semi-implicit model. This leads to a substantial economy of computer time.

The true test of the proposed integration scheme will consist in trying to use it in a complete multi-level atmospheric model. Such an experiment is underway at present and some preliminary results will soon become available.

References

Robert, A., J. Henderson and C. Turnbull, 1972: An implicit time integration scheme for baroclinic models of the atmosphere. Mon. Wea. Rev., Vol. 100, pp.329-335.

Robert, A., 1981: A stable numerical integration scheme for the primitive meteorological equations. Atmos. Ocean, Vol. 19, pp.35-46.

Robert, A., 1982: A semi-Lagrangian and semi-implicit numerical integration scheme for the primitive meteorological equations. J. Met. Soc. Japan, Vol. 60, pp.319-325.

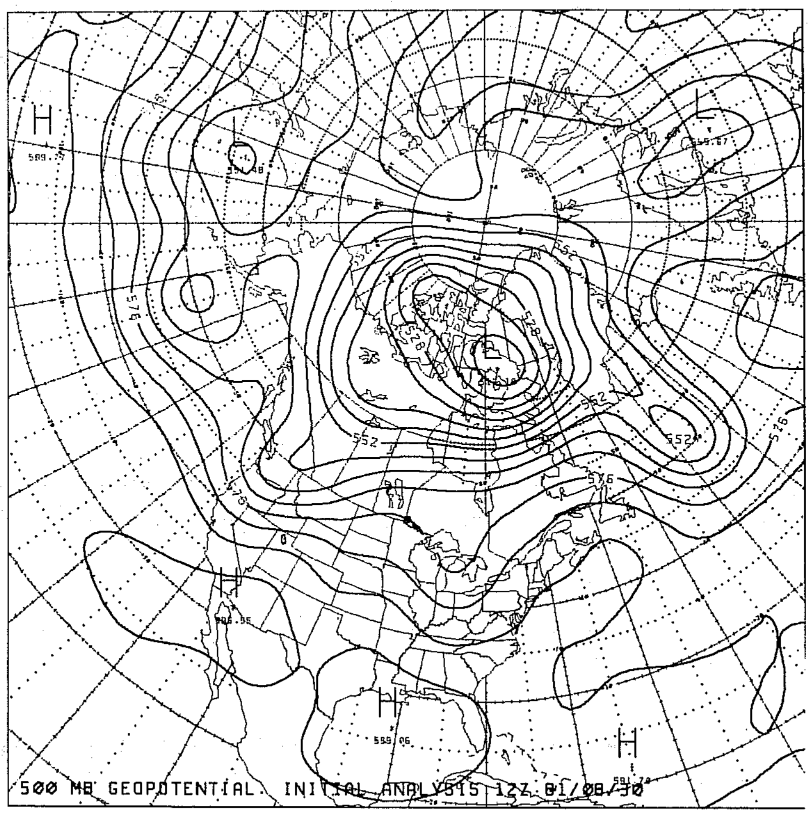

Fig. 1 Initialized 500mb geopotential at 12:00 GMT 30 August 1981.

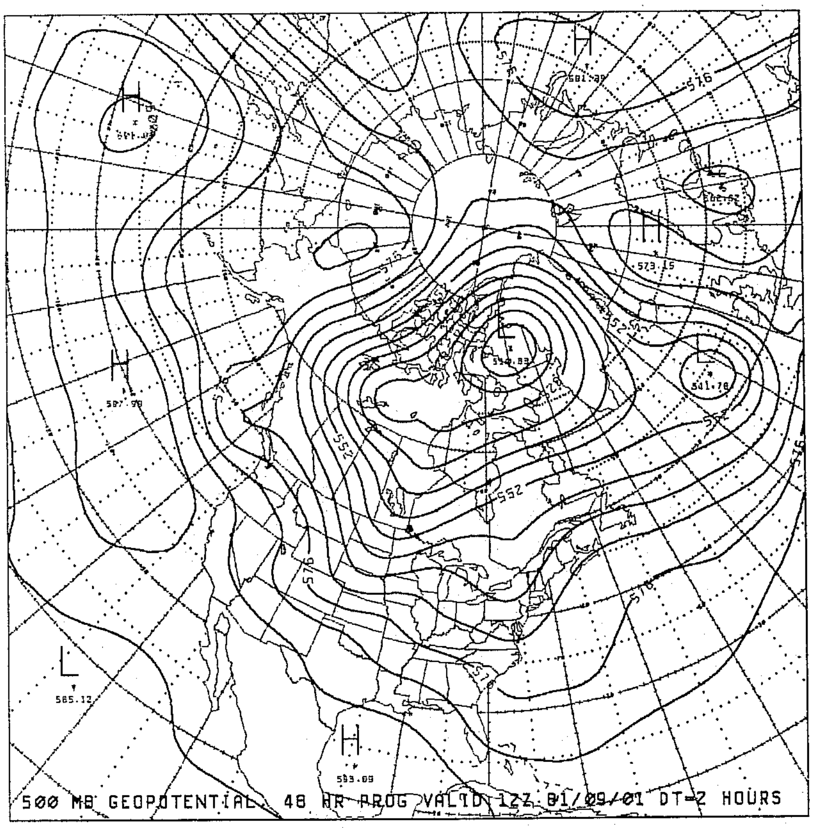

Fig. 2 48-hour forecast of the 500mb geopotential valid at 12:00 GMT 1 September 1981. Semi-Lagrangian and semi-implicit model with a two hour time step.

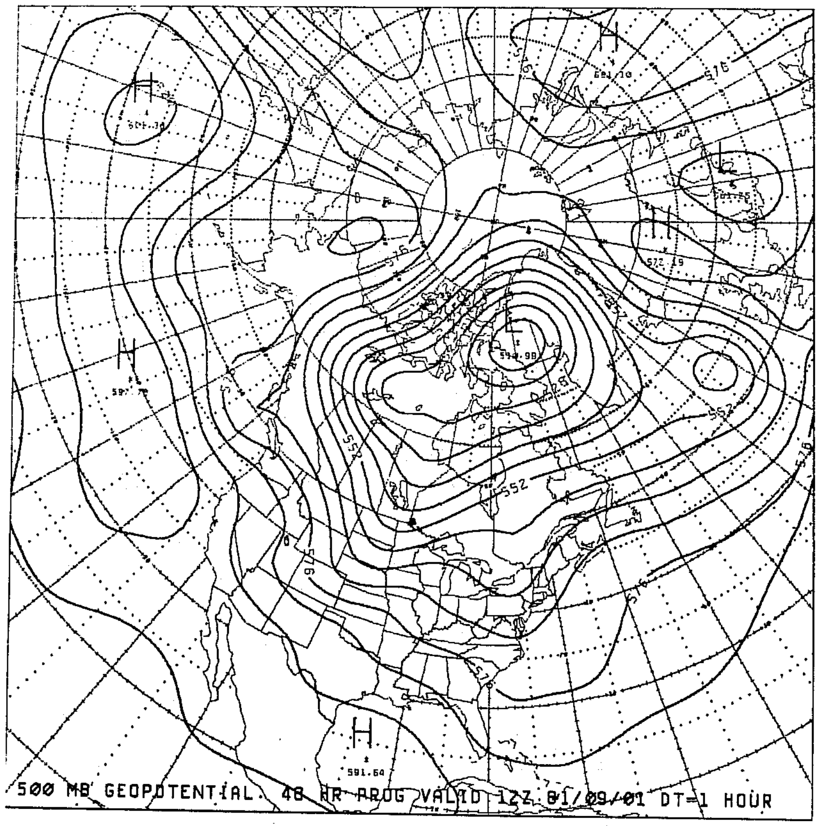

Fig. 3 48-hour forecast of the 500mb geopotential valid at 12:00 GMT 1 September 1981. Semi-Lagrangian and semi-implicit model with a one hour time step.