TECHNICAL MEMORANDUM

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Some properties of the Asselin time filter in the presence of friction

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Abstract

The stability and phase error per timestep of the amplitude response of physical, computational and exact modes have been investigated in the presence of friction, different diffusion schemes and various strengths of the Asselin (1972) time filter and propagation of the fluid, using a grid-point representation of the one-dimensional non-rotating linearized shallow water equations.

The analysis is similar to that by Schlesinger, Uccelini and Johnson (1983) (referred hereafter as SUJ) except for the inclusion of friction. Without filtering the effect of friction on modes is to damp them in comparison with the frictionless case. For diffusion and friction alone the effect of time filtering on the physical modes is almost negligible, while it is sufficient to suppress the computational mode. In this case, a heavy filter ($x_f \sim 1.0$) is as desirable as in the case of frictionless motion considered by SUJ.

In the case of diffusion, propagation and friction, the Asselin filter affects both the physical and computational modes, and the damping rate though the wave spectrum is not uniform.

For frictionless motion the damping rate of computational (physical) modes is smaller (larger) at intermediate wavelengths (as found by SUJ). In the presence of friction this property of Asselin filtering is enhanced, so that the computational mode is either larger than the physical one, or even unstable depending upon the strength of propagation, friction and time filtering.

For time filtering greater or equal to 0.5, the physical mode is found to be damped extensively in the presence of friction and strong propagation, so that the optimum value of Asselin filter 0.5 suggested by SUJ (frictionless case) does not seem to be optimal in this case.

An optimal Asselin filter parameter which counterpoints the combined results of both strong propagation and friction is found to be close to 0.3 for the Crank Nicholson diffusion scheme and for diffusion of the form $\nabla^4(\ )$, $\nabla^8(\ )$.

In this case there is no optimal parameter for the Lagged diffusion scheme.

Finally, although the computational mode of the Dufort-Frankel diffusion scheme is always larger than the physical one in some range of wavenumbers, an optimal value is close to 0.5.

1. INTRODUCTION

High frequency computational noise may appear in numerical models, either due to difficulties in finding initial conditions representative of the large scale atmospheric motions, or due to separation of solutions of alternative time steps when the leapfrog scheme is used (an example is demonstrated in Lilly, 1965). In the past, suppression of this noise was achieved by using certain finite-difference approximations for time which had a built-in damping property (Kurihara 1965).

Robert (1966) introduced a continuous time filter which was slightly modified and more comprehensively studied by Asselin (1972). Schlesinger, Uccelini and Johnson (1983, referred to as SUJ) used a one-dimensional linear analysis of the shallow-water wave equations on a non-rotating plane to investigate the effects of the Asselin time filter on both the stability and phase error of the leapfrog advection scheme using different diffusion schemes, and they found some optimal values of time filter depending upon the propagation of the fluid. These studies of the time filter did not take into account the effect of the Asselin time filter on stability in the presence of friction.

The purpose of this paper is to give some insight into the behaviour of the modulus of the amplification factor $|\phi|$ of physical, computational and exact modes, in the presence of friction for the one-dimensional linearized shallow water equations.

In Sect.2 a modified amplitude response of Asselin time filter is introduced when the friction is incorporated into the equations.

Sect. 3 contains a description of the one-dimensionnal linearized shallow-water wave equations in the presence of friction, the diffusion schemes and the modulus of amplification factors of physical, computational and exact solutions. In Sect.4 the results are presented. Finally in Sect.5 there are some concluding remarks.

2. AMPLITUDE RESPONSE OF THE ASSELIN TIME FILTER IN THE PRESENCE OF FRICTION

Consider a time series of the form

$$F^n = e^{i\omega n\Delta t} \quad , \quad n = 0,1,2,..... \tag{1}$$

where ω is a complex constant and n is a time level.

The Asselin time filter for the time series (1) is defined (Asselin 1972) by:

$$\bar{F}^n = F^n + \frac{1}{2} x_f \, (\bar{F}^{n-1} - 2F^n + F^{n+1}) \tag{2}$$

where the overbars denote filtered values and x_f is the filtering coefficient.

The response of the filter R' is defined by (Asselin 1972) as

$$\bar{F}^n = R'F^n \qquad R' = \phi \, \frac{2 + x_f(\phi-2)}{2\phi - x_f} \qquad \phi = e^{i\omega\Delta t} \tag{3}$$

As a simple representation of friction we include a Newtonian drag:

$$\frac{\partial F}{\partial t} = i\omega F - \lambda F \tag{4}$$

where λ is the frictional damping rate, which is taken as 0.0002 sec^{-1}. In this case the amplitude response R is given by:

$$R = \phi \, \frac{2 + x_f(\phi B-2)}{2\phi - x_f A} \tag{5}$$

where $A = e^{\int_{t-\Delta t}^{t} \lambda \Delta t}$ and $B = e^{-\int_{t}^{t+\Delta t} \lambda dt}$

If λ is constant in the time interval $[t-\Delta t, t+\Delta t]$, then A and B may be approximated by:

$$A = e^{\lambda^*} \text{ and } B = e^{-\lambda^*} \tag{6}$$

where $\lambda^* = \lambda\Delta t$ is a dimensionless frictional damping rate.

The effects of friction upon the stability and phase error are investigated here using $\lambda^* = 0.0$ and $\lambda^* = 0.06$ for the frictional damping rate.

In the case of no friction $\lambda^*=0$, from Eqn.(6) we obtain A=B=1, so that Eqn.(5) is identical to Eqn.(3). From Eqn.(5) it can be seen that the amplification factor and the phase shift produced by the filter are a function of the friction as well.

3. ONE DIMENSIONAL LINEARIZED SHALLOW WATER WAVE EQUATIONS IN THE PRESENCE OF FRICTION

Consider the one-dimensional linearized shallow water wave equations on a non-rotating plane with the friction, where for simplicity v' is assumed to be zero

$$\frac{\partial u'}{\partial t} = - U \frac{\partial u'}{\partial x} + i^{\mu-2} D \frac{\partial^{\mu} u'}{\partial x^{\mu}} - g \frac{\partial h'}{\partial x} + F_u \tag{7a}$$

$$\frac{\partial h'}{\partial t} = - U \frac{\partial h'}{\partial x} + i^{\mu-2} D \frac{\partial^{\mu} h'}{\partial x^{\mu}} - H \frac{\partial u'}{\partial x} + F_h \tag{7b}$$

where D is the diffusion coefficient,

u is the eastwards wind component,

h is the free surface height,

U and H are the eastwards velocity and the height of the of the basic state, which are constant,

$u' = U-u$,

$h' = H-h$,

μ is the order of linear dissipation

$F_u = -\lambda u'$ and

$F_h = -\lambda h'$.

Following an analysis similar to SUJ, Eqn.(7) has a wave type solution of the form

$$\begin{bmatrix} u' \\ h' \end{bmatrix} = \begin{bmatrix} \hat{u} \\ \hat{h} \end{bmatrix} e^{i(kx+\omega t)} \tag{8}$$

where $\hat{u}$, $\hat{h}$ are complex constants; k is the wavenumber (real) and ω is a complex frequency.

From Eqn.(7) using Eqn.(8) the following frequency relation may be obtained:

$$\omega = - k\ (U \pm C) + i(k^{\mu} D + \lambda) \tag{9}$$

where $C = (gH)^{\frac{1}{2}}$ (10)

Equation (9) shows that the wave propagates in both directions with speed $k(U \pm C)$ relative to basic flow.

Equations (7a) and (7b) are approximated by the following finite difference equations:

$$u_j^{n+1} - u_j^{n-1} = - \frac{U\Delta t}{\Delta x}(u_{j-1}^{n} - u_{j-1}^{n}) + i^{\mu-2} \frac{D\Delta t}{\Delta x^{\mu}}\ 2\ w u_j^{n+1} - \frac{g\Delta t}{\Delta x}\ (h_{j+1}^{n} - h_{j-1}^{n}) - 2(\lambda\Delta t)\bar{u}^{n-1} \tag{11a}$$

$$h_j^{n+1} - h_j^{n-1} = - \frac{U\Delta t}{\Delta x}(h_{j+1}^{n} - h_{j-1}^{n}) + i^{\mu-2} \frac{D\Delta t}{\Delta x^{\mu}}\ 2\ w h_j^{n+1} - \frac{H\Delta t}{\Delta x}\ (u_{j+1}^{n} - u_{j-1}^{n}) - 2(\lambda\Delta t)\bar{h}_j^{n-1} \tag{11b}$$

where primes have been dropped from u and h,

$x = j\Delta x\ (j=0, \pm 1, \ldots)$, $\quad t^n = n\Delta t\ (n=0,1,2\ldots)$,

$E^n_j = E(x,t^n)$ for any dependent variable E, and wP^n_j symbolizes, as in SUJ, any of the diffusion operators. The time derivatives are approximated by a leapfrog scheme using filtered values at time step n-1. In order to compare the results with SUJ the diffusion is formulated as in SUJ (i.e. Dufort-Frankel, Lagged and Crank-Nicholson schemes, referred hereafter as DF, L and CN respectively). Additionally, diffusion of the forms $\nabla^4 E^{n+1}$ and $\nabla^8 E^{n+1}$ (referred hereafter as d4, d8) are investigated.

Eqn.(11) have a wave-type solution of the form

$$\begin{bmatrix} u^n_j \\ h^n_j \end{bmatrix} = \begin{bmatrix} \hat{u} \\ \hat{h} \end{bmatrix} \phi^n e^{ik(j\Delta x)} \qquad (12)$$

A numerical solution of (11) is stable if $|\phi| \leq 1$. Using the same symbols as in SUJ Eqn.(11) may be written in matrix form as follows:

$$\begin{bmatrix} Q & 2iG^*\sin\sigma \\ 2iH^2\sin\sigma & Q \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{h} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (13)$$

where $U^* = \frac{U\Delta t}{\Delta x}$, $H^* = \frac{H\Delta t}{\Delta x}$, $G^* = \frac{g\Delta t}{\Delta x}$, $D^* = \frac{D\Delta f}{\Delta x}$, $\sigma = K\Delta x$,

$$\lambda^* = \lambda\Delta t, \qquad Q = \phi - R\phi^{-1} + 2iU^*\sin\sigma - 2i^{\mu-2}D^*\tilde{w} + 2\lambda^* R\phi^{-1}$$

and $\tilde{w}$ is a scheme-dependent multiplier applying to the diffusion w.

The existence of nontrivial solutions leads to relation

$$(AA\phi^2 + BB^+\phi + CC^+)(AA\phi^2 + \beta B^-\phi + CC^-) = 0 \qquad (14)$$

where $AA = 2\alpha$, BB^+, $BB^- = -x_f\alpha A + x_f\beta B + 2\gamma$ and CC^+, $CC^- = 2\beta - 2x_f\beta - x_f A\gamma$.

Signs $^+$ and $^-$ correspond to $V^* = U^* + C^*$ and $V^* = U^* - C^*$ respectively.

The multiplier $\tilde{w}$ and the parameters α, β and γ are defined as follows:

DUFORT-FRANKEL (DF)

$\tilde{w} = 2\cos\sigma - R\phi^{-1} - \phi$

$\alpha = 1 + 2i^{\mu-2}D^*$

$\beta = 2\lambda^* - 1 + 2i^{\mu-2}D^*$

$\gamma = 2iV^*\sin\sigma - 4i^{\mu-2}D^*\cos\sigma$

LAGGED (L)

$\tilde{w} = 2R\phi^{-1}(\cos\sigma - 1)$

$\alpha = 1$

$\beta = 2\lambda^* - 1 - 4i^{\mu-2}D^*(\cos\sigma - 1)$

$\gamma = 2i\ V^*\ \sin\sigma$

CRANK NICHOLSON (CN)

$\tilde{w} = (\phi + R\phi^{-1})(\cos\sigma - 1)$

$\alpha = 1 - 2i^{\mu-2}D^*(\cos\sigma - 1)$

$\beta = 2\lambda^* - 1 - 2i^{\mu-2}D^*(\cos\sigma - 1)$

$\gamma = 2i\sin\sigma V^*$

$\nabla^4(\ )^{n+1}$ (d4)

$\tilde{w} = (2\cos(2\sigma) - 8\cos\sigma + 6)\phi$

$\alpha = 1 - 2i^{\mu-2}D^*\tilde{w}\phi^{-1}$

$\beta = 2iV^*\sin\sigma$

$\nabla^8(\ )$ (d8)

$\tilde{w} = \phi(2\cos(46) - 16\cos(3\sigma) + 56\cos(2\sigma) - 112\cos\sigma + 70)$

$\alpha = 1 - 2i^{\mu-2}D^*\phi^{-1}\tilde{w}$

$\beta = 2\lambda^* - 1$

$\gamma = 2iV^*\sin\sigma$

Here $V^* = U^* \pm C^*$ and the positive sign is associated with the first factor in Eqn.(14).

As in SUJ, if (BB,CC) stand for any of (BB^+, CC^+), (BB^-, CC^-) from Eqn.(14), we obtain:

$$\phi = (-BB \pm \sqrt{BB^2 - 4AA.CC})/2AA \tag{15}$$

where the positive sign corresponds to physical modes and the negative to computational ones.

The phase error per time step for the physical and computational mode is defined, as in SUJ and Fromm (1969) by

$$Eph = \begin{cases} -\cos^{-1} [Re(\phi)/|\phi|]-\sigma V^* & Im(\phi)>0 \\ \cos^{-1} [Re(\phi)/|\phi|]-\sigma V^* & Im(\phi)<0 \end{cases} \tag{16}$$

Finally the amplitude response per time step of the exact mode is defined by

$$|\phi| = e^{-(\sigma^{\mu} D^*+\lambda^*)} \tag{17}$$

4. RESULTS

The effects of diffusion schemes, propagation of the fluid, friction, and time filter on physical, computational and exact modes have been investigated by considering various combinations of them.

The dimensionless frictional damping rate $\lambda^* = 0.06$ is taken to give a frictional damping rate of the order $\lambda = 0.0002\ sec^{-1}$.

The propagation is assumed to be moderate ($V^*=0.4$) when the mean velocity is $U=50\ msec^{-1}$ and the gravity velocity is $300\ msec^{-1}$, while for the strong propagation ($V^*=0.6$) the mean velocity is taken as $U=100\ msec^{-1}$.

In general the friction damps all the modes when it is applied at time step $t-\Delta t$, and when there is no filtering. In this case the modulus of the amplitude response of each mode is smaller than for the corresponding mode of the frictionless case.

4.1 Moderate diffusion and no propagation

From Fig.1 it can be seen that the Asselin time filter works very well in suppressing the computational mode, in spite of the incorporation of friction. As in the frictionless case (SUJ) high filter intensities are desirable ($x_f \sim 1.$) and the behaviour of all diffusion schemes is good. The phase error is seen to be independent of friction and there are some phase improvements for high filter intensities.

4.2 Moderate diffusion and moderate propagation

In this case, Fig.2, the Asselin time filter affects not only the computational mode, but also the physical one; moreover the damping rate through the wave spectrum is not uniform. At intermediate wavelengths, damping of the physical mode is a maximum, and damping of the computational mode is a minimum. This behaviour of heavy Asselin filtering is enhanced further when friction is incorporated into the model, so that the higher the propagation, friction and filtering the more pronounced is this property. Therefore, there is an upper limit to the ideal strength of the filter which is a function of the propagation of the fluid, friction and to some extent the diffusion; this upper limit is less than the limit for frictionless motion.

For all diffusion schemes an optimum value of x_f is found to be in the range (0.3, 0.5).

There are some pseudo-improvement of phase error, mainly at intermediate wavelengths, due to the fact that the physial and computational modes are damped differently as the filter strength increases.

4.3 Moderate diffusion and strong propagation

The friction causes a deterioration in the effect of time filtering on the modes so that not only does the damping rate through the wave spectrum become even more non-uniform, but also the physical mode is damped extensively at intermediate wavelengths as the filter strength increases. From Fig.3 it can be seen that for x_f=0.5 in the presence of friction, the physical mode has been damped extensively within the intermediate wavelengths, while in the frictionless case (SUJ) it has not. Therefore the filter strength x_f=0.5 suggested by SUJ is not an optimal one in this case.

Thus in the presence of friction, strong propagation and moderate diffusion an optimum value of x_f is close to 0.3 for d8, d4 and CN diffusion schemes. For the L diffusion scheme there is no optimum value, while an optimum value for DF close to 0.5 - although in this case the computational mode is greater than the physical one in some range of wavelengths.

4.4 Inclusion of friction at time step N

In Fig.4 are the results when the friction was incorporated into the linear model at time step n (case 2) rather than n-1, in the presence of strong propagation and moderate diffusion.

A comparison of Fig.4 with Fig.3 (case 1) reveals that:

a. In case 2 and in the presence of friction, the computational mode is unstable for x_f = 0. The Asselin filter works very well in suppressing the instability and for small filter intensities the scheme becomes stable.

b. For x_f = 0.3 or 0.5 the physical modes are almost identical in these two cases, while the computational mode is better suppressed in case 1.

c. For x_f = 0.75 the damping of the physical modes at intermediate wavelengths becomes more unrealistic in case 1, while the damping of the computational modes is almost identical.

Although in numerical models the physics is usually applied at time step n-1, it appears possible from these results that it can be applied at time step n together with time filtering.

5. GENERAL RESULTS

Without friction and filtering the modulus of computational and physical modes, for all diffusion schemes, are identical and close to unity, either throughout or in some range of the wave spectrum.

Without filtering and when the friction is applied at time step $t-\Delta t$, the effects of friction on modes is to damp them. For a light filter ($x_f \leq 0.12$) the computational mode is damped, but it is still almost of the same order of magnitude as the physical one. As the filter strength increases the effect of friction on modes depends on the propagation of the fluid.

Without propagation and in the presence of friction the damping rate of the computational mode is non uniform throughout the wave spectrum, and the computational mode is damped progressively. In this case higher values of filtering are required for all diffusion schemes, as in the frictionless case (SUJ).

In the presence of propagation and as the filter strength increases, the friction leads to a deterioration of the effect of the Asselin filtering on modes, and makes the computational mode more unstable and the physical one more stable in the range of intermediate wavelengths. Thus the physical mode departs from the exact one, while the computational mode becomes dominant or of the same order of magnitude as the physical one, and may even become unstable. An optimal Asselin filter parameter which counterpoints the combined results of both strong propagation and friction is found to be close to 0.3.

Finally, when the friction (and perhaps, by implication, more generally the physics) is applied at time step n, a stable scheme may be feasible using some higher values of filter intensity.

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Explanation of the Figures

All the figures consist of plots of the amplification factor $|\phi|$ per time step and phase error (Eph) as a function of normalised wave number ($\sigma=k\Delta x$). The abbreviation for the diffusion scheme is shown in the bottom left corner of each plot.

In each individual picture there are two plots for each mode, these corresponding to the cases of no friction ($\lambda^*=0.0$) and friction ($\lambda^*=0.06$).

In figures where the amplification factor is plotted, the dashed lines show the computational, the solid the physical and the dotted the exact mode. Usually the computational mode for $\lambda^*=0.06$ is labelled.

The hatched regions show areas where the modes are unstable.

The phase error is shown for the physical (solid line) and the computational mode (dotted lines). The y-axes of phase error has values from -3 to +3 rad Δt^{-1}. Columns from left to right corresponds to five different values of Asselin filter (x_f = 0.0, 0.3, 0.5, 0.75 and 1.0).

Moderate diffusion and no propagation

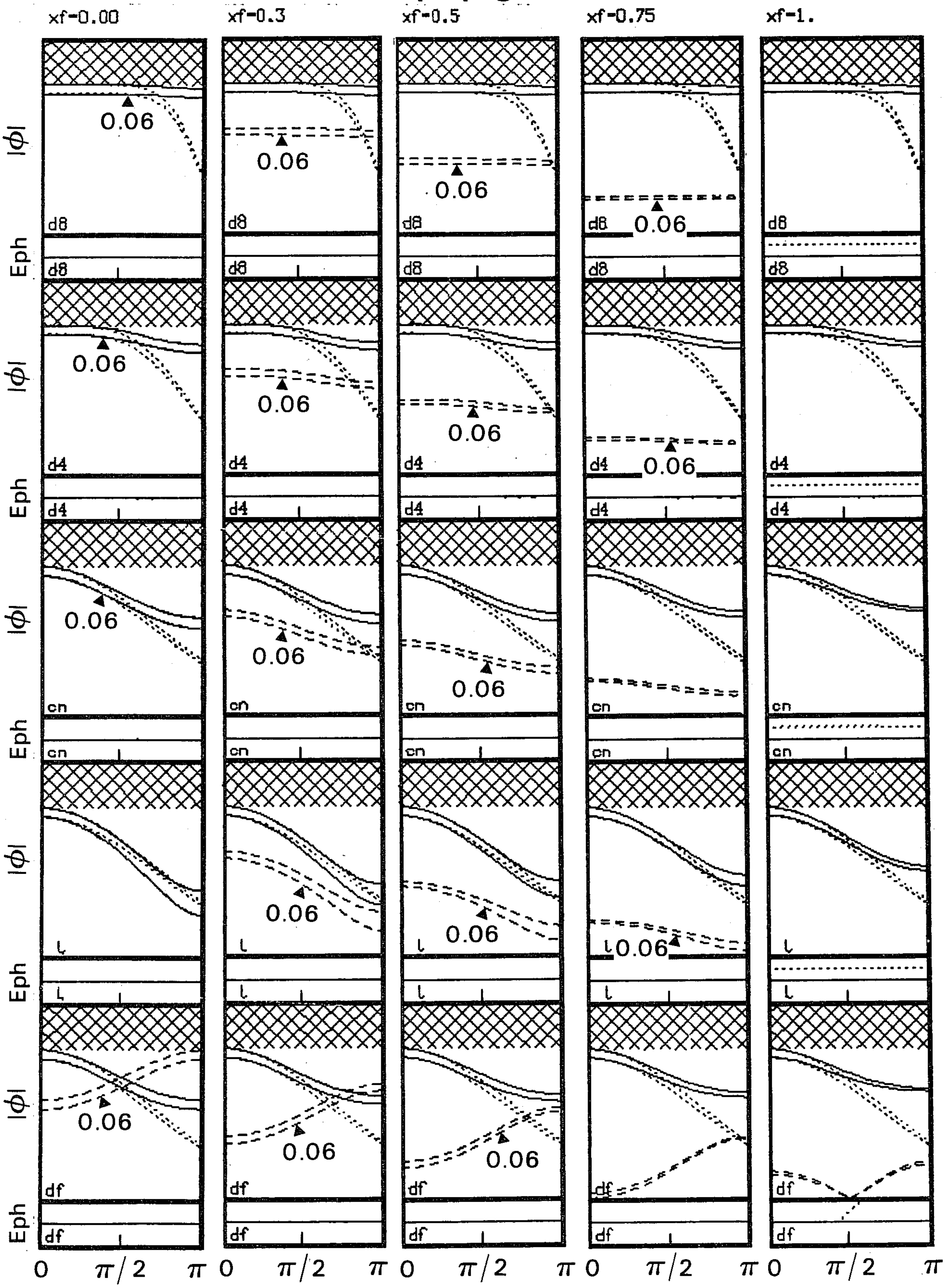

Fig. 1 Plot of amplification factor (|φ|) per time step and phase error (Eph) as a function of normalised wavenumber ($\sigma = k\Delta x$) for moderate diffusion and no propagation (See the opposite page for an explanation of the layout of the diagrams.)

Moderate diffusion and moderate propagation

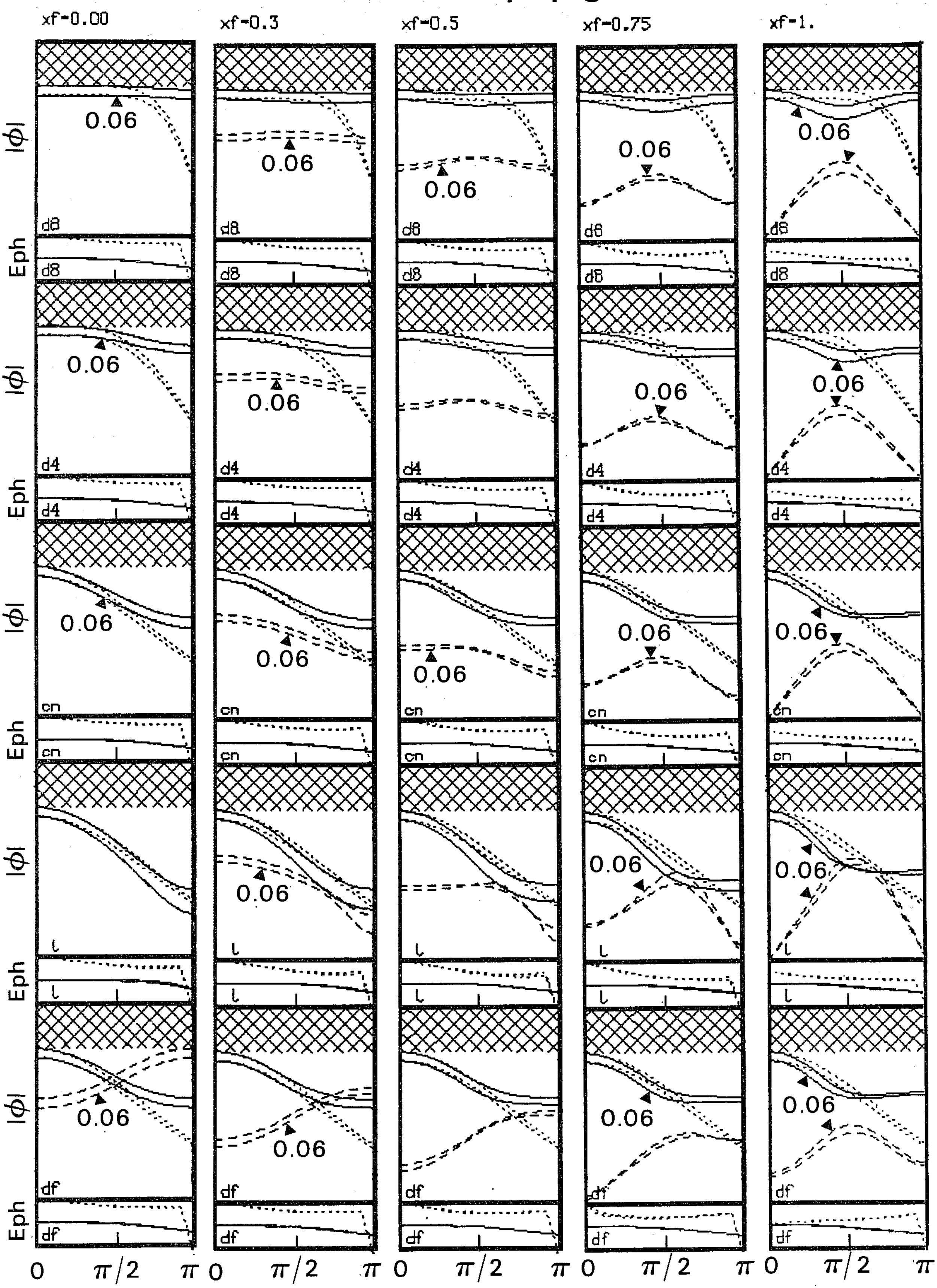

Fig. 2 As in Fig. 1, except that the propagation is moderate.

Moderate diffusion and strong propagation

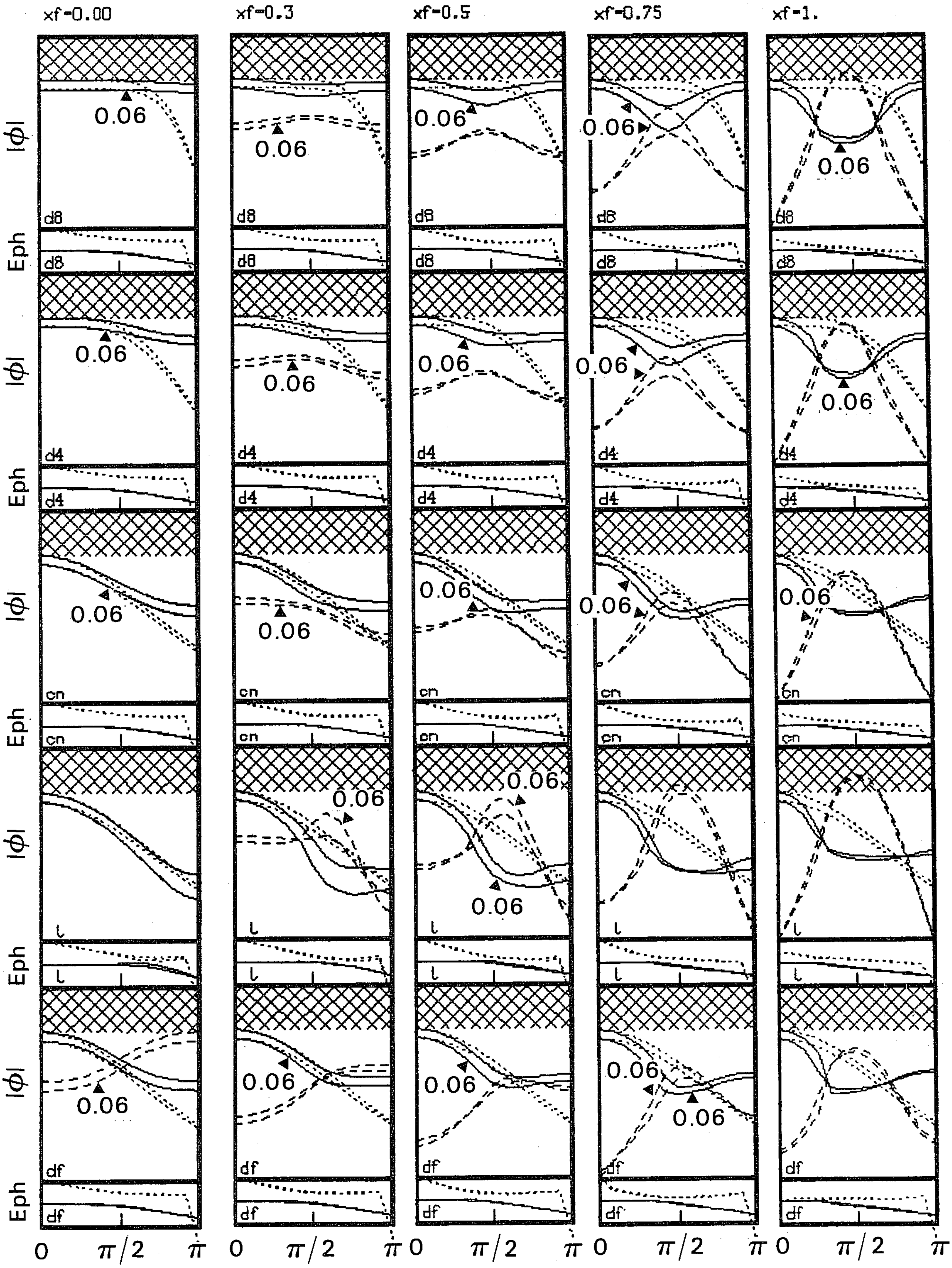

Fig. 3 As in Fig. 1, except that the propagation is strong.

Moderate diffusion and strong propagation

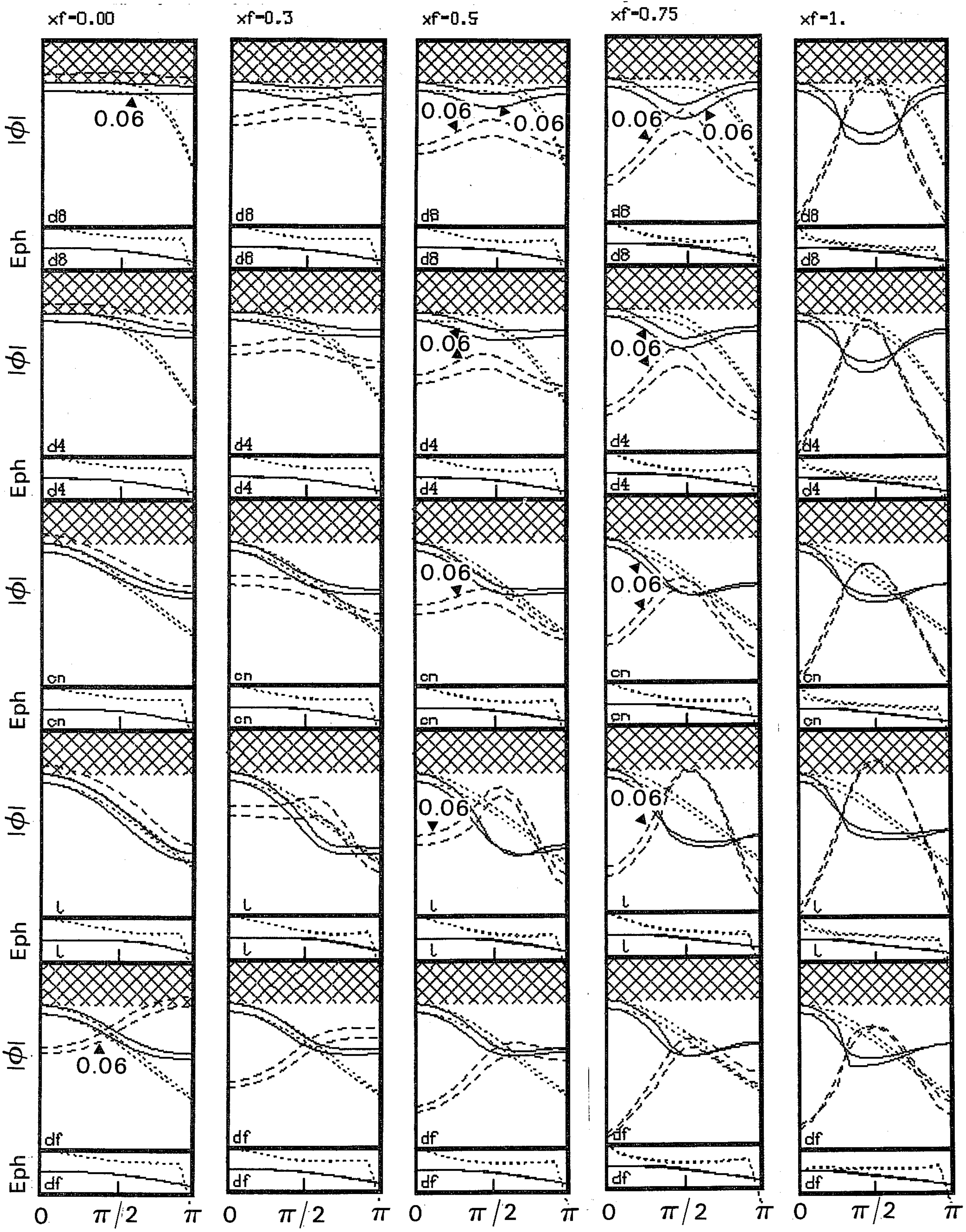

Fig. 4 As in Fig. 3, except that the friction was applied at time step n.