IMPLEMENTATION OF SEMI-LAGRANGIAN ADVECTION IN THE NEXT GENERATION U.K. MET. OFFICE UNIFIED MODEL.

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Abstract

This paper describes the semi-Lagrangian scheme as it has been implemented in the semi-implicit non-hydrostatic version of the Unified Model currently being developed at the U.K. Met Office. The implementation has raised various questions and these are discussed although not necessarily answered.

1 INTRODUCTION

The current U.K. Met Office Unified Model is a hydrostatic grid-point Eulerian model using the Arakawa B-grid in the horizontal and the Lorenz vertical grid-staggering. The equations of motion are solved using a split-explicit treatment for the gravity wave terms, and an explicit Heun finite-difference scheme for the advective terms. The Heun scheme can be chosen to be either second or fourth order accurate in space in the horizontal, but is only first order accurate in space in the vertical for unequally spaced grids. To avoid the severe timestep limitations incurred near the poles when schemes of this type are implemented on equally spaced latitude/longitude grid on the sphere, Fourier filtering is performed on the increments to ensure no unstable modes are allowed into the model fields. A full description of the scheme can be found in Cullen & Davies (1991). This scheme has several undesirable properties for advection of meteorological fields, namely;

- The severe restriction on the maximum timestep for which the scheme is stable, or the need to use Fourier filtering to allow a larger timestep to be used.
- The restriction to first order accuracy in space on unequally spaced grids. This produces inaccuracies in the vertical transport since the grid is typically highly unequally spaced in the vertical.
- The severe over-shooting and under-shooting of the scheme make it a poor choice, particularly for moisture fields where negative values cannot be allowed to exist. The scheme can be improved in this regard by adding artificial diffusion, but this is an undesirable addition purely to control numerical noise.

While it is possible to address some of these issues whilst keeping the basic Eulerian framework, all these problems can be avoided by using a semi-Lagrangian advection scheme, provided a monotone interpolation scheme is used. When considering the design for the new non-hydrostatic Unified Model it was decided to use a two time-level semi-Lagrangian advection scheme coupled with a semi-implicit treatment of the gravity waves as the basis for the scheme. The new model is a fully-compressible non-hydrostatic model on an Arakawa C-grid in the horizontal and uses the Charney-Phillips grid in the vertical, see figure 1 for a comparison of the Charney-Phillips and Lorenz grids for a non-hydrostatic model. The reasoning behind the choices made for the new model can be found in Cullen, Davies, Mawson, James & Coulter (1995). In the next section the solution procedure for the new model is briefly described, and in the section thereafter the implementation of the semi-Lagrangian scheme is presented along with a discussion of the issues that have arisen during implementation and testing.

Upper Boundary	$r,\eta,\theta,w,\dot{\eta}$	k=N	$r,\eta,w,\dot{\eta}$
	r,η,ρ,p,u,v	k=N-0.5	r,η,ρ,θ,p,u,v
	.		.
	.		.
	.		.
	$r,\eta,\theta,w,\dot{\eta}$	k=1.0	$r,\eta,w,\dot{\eta}$
	r,η,ρ,p,u,v	k=0.5	r,η,ρ,θ,p,u,v
	$r,\eta,\theta,w,\dot{\eta}$	k=1.0	$r,\eta,w,\dot{\eta}$
	r,η,ρ,p,u,v	k=0.5	r,η,ρ,θ,p,u,v
surface	$r,\eta,w,\dot{\eta}$	k=0	$r,\eta,w,\dot{\eta}$

Figure 1: Vertical grid staggerings: Charney-Phillips on left, Lorenz on right.

2 PROPOSED NON-HYDROSTATIC MODEL

For simplicity we present only the dry version of the algorithm. The dry 3D non-hydrostatic equations are

$$\frac{D\underline{u}}{Dt} = -f\underline{k} \times \underline{u} - C_p\theta\nabla\Pi - g\underline{k} + \underline{F} \tag{1}$$

$$\frac{D\theta}{Dt} = S \tag{2}$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{\partial r/\partial \eta}\left[\frac{1}{cos\phi}\frac{\partial}{\partial \lambda}\left(\frac{\rho u}{r}\frac{\partial r}{\partial \eta}\right) + \frac{1}{cos\phi}\frac{\partial}{\partial \phi}\left(\frac{\rho v cos\phi}{r}\frac{\partial r}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\rho\dot{\eta}\frac{\partial r}{\partial \eta}\right)\right] = 0 \tag{3}$$

$$p = \Pi\theta\rho R \tag{4}$$

where D/Dt is the Lagrangian derivative, $\underline{u} = (u, v, w)$, the density ρ contains a factor of r^2, S and $\underline{F}$ are forcing functions, Π is the Exner pressure, r is the distance from the centre of the earth, η is a normalized hybrid height co-ordinate such that $\eta = 0$ at the surface and $\eta = 1$ at the upper boundary, and

$$\dot{\eta} = \frac{1}{\partial r/\partial \eta}\left[w - \frac{u}{r cos\phi}\frac{\partial r}{\partial \lambda} - \frac{v}{r}\frac{\partial r}{\partial \phi}\right] \tag{5}$$

$$\nabla\Pi = \left(\frac{1}{r cos\phi}\left.\frac{\partial \Pi}{\partial \lambda}\right|_r, \frac{1}{r}\left.\frac{\partial \Pi}{\partial \phi}\right|_r, \frac{1}{r}\frac{\partial \Pi}{\partial r}\right) \tag{6}$$

where $|_r$ denotes a horizontal derivative taken along a constant r surface.

The solution algorithm is as follows:

Step 1: Use equations (1) and (2) to produce estimates of $\underline{u}$ and θ at time level $n+1$ using the time level n values only and the semi-Lagrangian advection scheme.

Step 2: Require that the equation of state holds at time level $n+1$. Define corrections $X' = X^{n+1} - X^n$ where X is any of the data variables, u, v, w, Π, θ or ρ, and write the equation of state in terms of the values at time level n and the corrections. The procedure is to then linearize the equation of state in terms of the corrections and to write all the corrections in terms of Π'. This results in a variable coefficient Helmholtz equation for Π' which is solved via an iterative method, for example a pre-conditioned conjugate gradient method. Some compromises have to be made in forming the corrections and the result of this is that we obtain an estimate of Π' rather than the exact value. The correction to Π is then used to calculate $\underline{u}'$.

Step 3: Calculate p^{n+1} from $\Pi^n + \Pi'$ and ρ^{n+1} from the continuity equation (3). These terms agree exactly with those implied in the creation of the Helmholtz equation. In the incompressible case this ensures exact removal of the mass divergence.

Step 4: Calculate $\underline{u}^{n+1}$ and θ^{n+1} using equations (1) and (2) and the semi-Lagrangian advection scheme with the advecting variables and trajectory calculated using an approximation to $\underline{u}^{n+1/2}$ given by $\underline{u}^n + \alpha\underline{u}'$, where ideally the time-weight $\alpha = 0.5$. This step is not consistent with the way these terms are treated in the formation of the Helmholtz equation, and as a result the equation of state is not exactly satisfied at time level $n+1$.

This procedure can be understood by considering a much simpler problem which explains in essence what the algorithm is doing. Consider simply

$$\frac{DQ}{Dt} = 0$$

Lagrangian advection of some quantity Q. Given the location of a particular blob of Q at time t, then to find the position at time $t+\Delta t$ it is necessary to calculate the trajectory it followed, and to do this the velocity field over the time interval $(t, t+\Delta t)$ must be known. Given this velocity field the advection equation can be solved. This is essentially what the integration scheme is doing, except that it is first necessary to calculate the velocity field over the time interval. The first two steps of the algorithm provide a good estimate to the velocity field at time $n+1$ and this is used along with the value of the velocity field at time n to give the velocity field over the entire time interval. The advection equations are then solved using this velocity field. The first two steps only provide a good estimate, not an exact answer, because of the difficulty in solving step 2 exactly. In principle the procedure could be iterated to obtain improved estimates of the velocity field at time-level $n+1$, but as the error in the estimate is probably not significant compared to other model errors it is unlikely any significant benefit would accrue from this.
Note that in this solution procedure the only fields which are advected are $\underline{u}$ and θ, and that the continuity equation is solved in the flux form.

3 SEMI-LAGRANGIAN SCHEME

In designing the semi-Lagrangian scheme there are several areas which need considering namely;

- The advection of the wind field
- Choice of departure point scheme
- How to calculate the departure point when it lies outside the model domain
- What order of interpolation to use

- How to enforce monotonicity
- How to ensure conservation, in the sense that the integral over the whole model domain of the quantity being advected remains constant

Each of these issues is now examined in turn and their implementation is described. Various questions and concerns that have arisen during this process are also discussed.

3.1 Advection of the wind field.

The velocity field is advected as a vector, (Bates 1988, Côté 1988, Ritchie 1988) and subsequent papers, which avoids the problems associated with the polar singularity. Writing equation (1) in its component form and denoting by subscript a a value at the arrival point, and subscript d a value at the departure point gives

$$u_a - \Delta t \alpha F_{ua} = u_d + \Delta t(1-\alpha)F_{ud} \quad (7)$$

$$v_a - \Delta t \alpha F_{va} = v_d + \Delta t(1-\alpha)F_{vd} \quad (8)$$

$$w_a - \Delta t \alpha F_{wa} = w_d + \Delta t(1-\alpha)F_{wd} \quad (9)$$

where (F_u, F_v, F_w) represent the terms on the right-hand-side of equation (1) for each component respectively, Δt is the timestep, and α a time-weight with $\alpha = 1$ giving a fully-implicit scheme and $\alpha = 0$ a fully-explicit scheme. These equations are now written in terms of the local unit vectors at the arrival and departure points respectively giving

$$(X_u\underline{i} + X_v\underline{j} + X_w\underline{k})_a = (Y_u\underline{i} + Y_v\underline{j} + Y_w\underline{k})_d \quad (10)$$

where (X_u, X_v, X_w) and (Y_u, Y_v, Y_w) denote the left-hand-sides and right-hand-sides of equations (7)-(9) respectively. Co-ordinate geometry now shows that

$$\begin{aligned} X_u &= cos(\lambda_a - \lambda_d)\, Y_u \\ &+ sin\phi_d\, sin(\lambda_a - \lambda_d)\, Y_v \\ &- cos\phi_d\, sin(\lambda_a - \lambda_d)\, Y_w \end{aligned} \quad (11)$$

$$\begin{aligned} X_v &= -sin\phi_a sin(\lambda_a - \lambda_d)\, Y_u \\ &+ (cos\phi_a cos\phi_d + sin\phi_a sin\phi_d cos(\lambda_a - \lambda_d))\, Y_v \\ &+ (cos\phi_a sin\phi_d - sin\phi_a cos\phi_d cos(\lambda_a - \lambda_d))\, Y_w \end{aligned} \quad (12)$$

$$\begin{aligned} X_w &= cos\phi_a sin(\lambda_a - \lambda_d)\, Y_u \\ &+ (sin\phi_a cos\phi_d - cos\phi_a sin\phi_d cos(\lambda_a - \lambda_d))\, Y_v \\ &+ (sin\phi_a sin\phi_d + cos\phi_a cos\phi_d cos(\lambda_a - \lambda_d))\, Y_w \end{aligned} \tag{13}$$

To solve these equations it is necessary to find the departure points of the trajectories, interpolate the quantities on the left-hand-sides to them, and evaluate the quantities on the left-hand-side which depend on the values of the fields at the end of the timestep, time-level $n+1$. In the integration procedure described in the previous section this is done in three stages. First the equations are solved using trajectories calculated from the wind field at the current time level n, and evaluating the terms on the left-hand-side using the fields at time level n. This estimate of the solution at time-level $n+1$ is used in the Helmholtz equation which when solved gives an improved estimate to $\underline{u}^{n+1}$. This improved estimate is then used to re-calculate the solution to the equations using the wind field at the mid-time level, given by linear interpolation in time between the value at time-level n and the improved estimate at time-level $n+1$, to find the departure points. The improved estimates to the fields at time-level $n+1$ are also used to evaluate the terms on the left-hand-side of equations (11) to (13). Currently to reduce computational cost the following measures are being used;

- w at time level $n+1$ is set equal to the improved estimate to w at time level $n+1$ is used rather than re-calculating the solution to equation (13).
- The forcing terms (F_u, F_v, F_w) at the departure point are not re-calculated. This leads to a loss in temporal accuracy if the departure point of the trajectories given by the wind field at time-level n and the estimate at time level $n+1/2$ differ significantly.
- Linear interpolation could be used to interpolate the fields to the departure point for the first solution of the equations rather than the current cubic interpolant, if this is found not to degrade the solution significantly.

3.2 Departure point scheme

Semi-Lagrangian departure point calculations have so far been mainly based on the work of Robert (1981) and consist of finding the departure point of a trajectory given the arrival point, these trajectories are referred to as upstream or backward trajectories. An alternative approach to formulate the model in terms of forward or downstream trajectories, as described in Leslie & Purser (1995), is not considered here and hence is omitted from this discussion. Two schemes based on the work of Robert (1981) have been implemented and tested, one as described in McDonald & Bates (1987), henceforth referred to as Robert, and a two-time level version of the scheme described in Ritchie & Beaudoin (1994), henceforth referred to as Ritchie. An alternative way of finding the departure point was proposed by McGregor (1993), henceforth referred to as McGregor, and this has also been implemented. The first two schemes seek to find the velocity field at the mid-point in time and space of the trajectory which the fluid follows and use that to calculate the departure point. This procedure requires interpolations to be performed to find the velocities at the mid-point in space and can be expensive if high order interpolation is used. This method can also cause problems on massively parallel distributed memory computers, since near the pole this calculation can cause non-local inter-processor communication to occur. The version of the schemes used here use one iteration of the procedure to find the mid-point of the trajectory in space and linear interpolation of the velocities to the mid-point of the trajectory. The McGregor scheme uses a truncated Taylor series to estimate the departure point and approximates the derivatives therein by finite-differences. This has potential advantages in that it is cheaper to compute and requires only local communication on a massively parallel distributed memory computer. The version of the scheme used here truncates the Taylor series after 4 terms.

To investigate the performance of the three schemes, and also to investigate the performance of different interpolation schemes, test problem 1 from Williamson, Drake, Hack, Jakob & Swarztrauber (1992), advection of a cosine bell under solid body rotation, was used. The problem is as described in Williamson et al. (1992) except that the angles of the solid body rotation relative to the equator are chosen to be 0^o, 45^o and 90^o rather than 0^o, 15^o and 90^o. The code was run as a two-level model with no vertical velocity and the same specification of the problem for each level, purely to avoid re-writing the interpolation and departure point codes as two-dimensional rather than three-dimensional codes. The code was run on one processor of a Cray C-90 and the timings show the total cost per timestep of the code for rotation at angle 0^o only, since the timings for the other angles differ from them by only a small random amount. The results are shown in tables 1-5. The normalised L_2 error, denoted Norm Error in the tables, is defined as in Williamson et al. (1992) by

$$L_2(h) = \frac{(I[(h-h_T)^2])^{1/2}}{(I[h_T^2])^{1/2}}$$

where $I(h)$ is the discrete approximation to the integral over the sphere given analytically by

$$I(h) = \frac{1}{4\pi}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} h(\lambda,\phi)\,d\phi\,d\lambda$$

for spherical polar co-ordinates (λ, ϕ), with h_T the true solution.

Comparing all the angles and all the interpolation scheme in tables 1-5 for the schemes of Robert and Ritchie there are no significant differences in accuracy, and no systematic benefit of one scheme over the other. The only noticeable difference is in the CPU cost with the Ritchie scheme significantly cheaper. Comparing the Ritchie scheme with that of McGregor there are no significant differences except for flow at 90^o. For this direction the McGregor scheme is marginally worse for the basic Lagrange interpolation schemes, but better for the quasi-Lagrange schemes, (see section 3.4 for details of these schemes). The McGregor scheme is also cheaper than the Ritchie scheme, but the saving is less significant than between the Robert and Ritchie schemes. From this evidence it would seem sensible to use the McGregor scheme as it is the cheapest scheme and the performance in terms of accuracy of all the schemes is comparable. However there is a drawback with the three-dimensional version of the scheme as described in McGregor (1993), namely there is a systematic error in the vertical position of the departure point. For example, if $v = w = 0$ then the Taylor series truncated after one term gives an error in the vertical of

$$R\left((1 + (\nu\Delta\lambda)^2)^{1/2} - 1\right)$$

where R is the distance of the departure point from the centre of the earth, ν is the Courant number and $\Delta\lambda$ the grid-length in radians. For a Courant number of 1 and a resolution of 3.75^o then this error is approximately 10 meters when the Taylor series is truncated after 4 terms. Whilst this error is relatively small at the levels where such Courant numbers occur, it is the fact that it is always positive and hence systematic that is of concern. The effect of this error has yet to be examined by us in a complete three-dimensional model. It is possible the scheme could be modified so that this problem is removed, but as yet this has not been attempted.

3.3 Trajectories outside the model domain

Consider the problem of a trajectory which goes below the bottom level where data is held for the field required at the departure point. This can easily happen since upward motion at the bottom model level for any quantity implies that the trajectory originated below the

Scheme	angle	Max Height	min Height	Norm Error	Max Error	Min Error	CPU/ timestep
Robert	0	944.34	-41.44	0.098	55.65	-68.13	0.0552
Robert	45	978.04	-24.24	0.058	33.51	-24.24	-
Robert	90	996.93	-19.50	0.036	19.50	-21.98	-
Ritchie	0	944.34	-41.44	0.098	55.65	-68.13	0.0502
Ritchie	45	978.44	-24.25	0.059	33.24	-38.30	-
Ritchie	90	996.93	-19.50	0.032	19.50	-21.58	-
McGregor	0	944.34	-41.44	0.098	55.65	-68.12	0.0487
McGregor	45	978.50	-24.24	0.058	31.24	-37.53	-
McGregor	90	992.12	-19.47	0.047	63.77	-61.29	-

Table 1: Cosine bell problem using quintic Lagrange interpolation

Scheme	angle	Max Height	min Height	Norm Error	Max Error	Min Error	CPU/ timestep
Robert	0	740.69	-49.74	0.313	259.30	-209.68	0.0307
Robert	45	723.09	-31.52	0.267	276.90	-134.24	-
Robert	90	877.78	-35.82	0.164	122.21	-113.69	-
Ritchie	0	740.69	-49.74	0.313	259.30	-209.68	0.0257
Ritchie	45	724.62	-31.34	0.267	275.37	-134.47	-
Ritchie	90	877.78	-35.82	0.162	122.21	-113.69	-
McGregor	0	740.69	-49.74	0.313	259.30	-209.68	0.0240
McGregor	45	724.73	-31.37	0.267	275.26	-133.42	-
McGregor	90	872.79	-36.62	0.165	133.25	-151.38	-

Table 2: Cosine bell problem using cubic Lagrange interpolation

Scheme	angle	Max Height	min Height	Norm Error	Max Error	Min Error	CPU/ timestep
Robert	0	740.69	-49.74	0.313	259.30	-209.68	0.0271
Robert	45	812.93	-43.33	0.214	187.06	-147.39	-
Robert	90	877.78	-45.10	0.135	122.21	-113.69	-
Ritchie	0	740.69	-49.74	0.313	259.30	-209.68	0.0220
Ritchie	45	814.31	-43.10	0.215	185.68	-147.63	-
Ritchie	90	877.78	-45.04	0.131	122.21	-113.69	-
McGregor	0	740.69	-49.74	0.313	259.30	-209.68	0.0204
McGregor	45	814.46	-43.13	0.215	185.53	-146.25	-
McGregor	90	872.79	-46.84	0.135	133.25	-151.38	-

Table 3: Cosine bell problem using quasi-cubic Lagrange interpolation

Scheme	angle	Max Height	Norm Error	Max Error	Min Error	CPU/ timestep
Robert	0	552.31	0.415	447.68	-209.35	0.0340
Robert	45	637.66	0.318	362.33	-142.49	-
Robert	90	712.34	0.231	287.65	-134.79	-
Ritchie	0	552.31	0.415	447.68	-209.35	0.0287
Ritchie	45	638.73	0.319	361.29	-142.67	-
Ritchie	90	712.29	0.229	287.70	-134.79	-
McGregor	0	552.34	0.415	447.65	-209.35	0.0280
McGregor	45	639.00	0.318	360.99	-141.68	-
McGregor	90	772.39	0.212	227.60	-164.29	-

Table 4: Cosine bell problem using monotone cubic Lagrange interpolation

Scheme	angle	Max Height	Norm Error	Max Error	Min Error	CPU/ timestep
Robert	0	552.31	0.449	447.68	-208.44	0.0286
Robert	45	505.29	0.417	494.70	-140.41	-
Robert	90	709.55	0.235	290.44	-135.05	-
Ritchie	0	552.31	0.449	447.68	-208.44	0.0237
Ritchie	45	507.49	0.415	492.50	-140.86	-
Ritchie	90	709.55	0.237	290.44	-135.05	-
McGregor	0	552.31	0.449	447.68	-208.44	0.0221
McGregor	45	507.60	0.414	492.39	-149.97	-
McGregor	90	721.65	0.235	278.34	-167.37	-

Table 5: Cosine bell problem using monotone quasi-cubic Lagrange interpolation

bottom level. The question is, how should the departure point of the trajectory be defined to best estimate the field below the bottom level, or should the departure point be allowed to lie outside the model domain and the value at the point obtained by extrapolation ? If horizontal advection term dominates the vertical advection term then constraining the trajectory to lie on the bottom model level will be a good approximation. This approach is typical of semi-Lagrangian schemes in use today, for example at ECMWF, (Ritchie, Temperton, Simmons, Hortal, Davies, Dent & Hamrud 1995). An alternative to this approach would be to extrapolate the quantities required below the bottom boundary, which would include the wind fields if using the departure point schemes based on the work of Robert (1981). The effect of doing this we have yet to investigate, but would depend on being able to use extrapolation which is consistent with the variation in the fields, for example linear if the fields vary linearly with height. One way of investigating this would be compare simulations of a known problem, such as the cold gravity current (Carpenter, Droegemeier, Woodward & Hane 1990), at different vertical resolutions and using different ways of limiting the trajectory compared to estimating the value at the departure point. So far only trajectories which leave the domain because of the vertical velocity have been considered, a more difficult question is what to do with trajectories which end up inside orography. Setting the departure point to be at the lowest model level could produce serious errors in stratified flow around orography where the departure point could be spuriously moved to the top of a ridge when, for example, it should lie in a valley. This problem is probably not serious in the current global forecast models but could be of concern in high resolution mesoscale models. For departure points which lie inside orography the approach of leaving them where they are calculated to be and extrapolating the fields could lead to large errors, and is probably not a viable solution. The question is whether there is a better solution than simply moving the point to the lowest model level ? It might be feasible to examine the size of the error in the vertical displacement caused by moving the point to the lowest model and use that to decide if the error in the position of the departure point is likely to be horizontal or vertical. If the error is small, a couple of meters or less, then is likely to be due to the vertical velocity and either moving it to the bottom model level or extrapolation will produce a good solution. If the error is large then is likely due to an error in the horizontal position and some other approach should be used, for example moving the point in the horizontal or horizontal extrapolation. Currently we do not know what the impact of such a change would be or how frequent does the problem it seeks to alleviate occur. At the moment we simply reset any trajectories which go below the bottom model level to the value at the bottom model level for that field.

3.4 Interpolation

The choice of interpolation scheme to use to evaluate fields at the departure point depends on how important the truncation error of the advective terms is relative to other model errors. It could be argued that if you are using second order centred finite-differences to evaluate derivatives on the right-hand-side of the momentum equation (1) that a second order accurate interpolation scheme would be appropriate. The majority of semi-Lagrangian schemes at the moment use a tri-cubic interpolant of some kind. A popular choice has been a modified tri-cubic scheme where some of the one dimensional cubic interpolants which make up the tri-cubic interpolant are replaced by linear interpolation to reduce the cost. Examples of this type of scheme can be found in Bates, Moorthi & Higgins (1993) and Ritchie et al. (1995). Linear, cubic and quintic Lagrange interpolation plus the ECMWF quasi-cubic Lagrange interpolant (Ritchie et al. 1995) have been implemented along with monotone versions of each of them; how to enforce monotonicity is considered in section 3.5. To compare them the cosine bell problem as described in section 3.2 was used, the results for quintic Lagrange, cubic Lagrange, ECMWF quasi-cubic Lagrange, ECMWF monotone cubic Lagrange, monotone quasi-cubic Lagrange are given in tables 1 to 5 respectively. The results for the two non-monotone cubic based schemes for rotation along the equator, angle of 0^0, are identical as are the results for the two monotone schemes. The results for the non-monotone schemes are better in terms of the maximum height of the cone and the error norm, but not in terms of the minimum value. Similar results apply for flow straight across the pole, angle of 90^o, except that the monotone quasi-cubic scheme is marginally worse than the monotone cubic scheme, purely because they use a different way of implementing monotonicity. It is at 45^o that a significant difference in the results is noticeable. Here the quasi-cubic has significantly better scores than the cubic, error norm of 0.215 compared to 0.267, but conversely the monotone quasi-cubic has significantly worse scores than the monotone cubic, error norm of 0.414 compared to 0.318, and this time it is not purely due to using a different way of implementing monotonicity. The quasi-cubic scheme is giving better results by producing bigger overshoots than the cubic scheme, and hence the worse results when the overshoots are prevented by enforcing monotonicity. The question is, what type of interpolation scheme should be used ?

This question has arisen in the wider mathematical community in computational fluid dynamics (CFD) with the consensus that schemes which produce no new maxima or minima should be used, see for example Garcia-Navarro & Priestley (1994), referred to as monotone schemes here but which exist under a variety of different names in the CFD community. Such schemes would naturally be used for tracers and moisture variables in atmospheric models to prevent spurious negative values. They should also be used for momenta to prevent spurious maxima and minima which will probably appear as spurious kinetic energy sources. The question that remains is how to enforce monotonicity ?

3.5 Monotonicity

The conditions for monotonicity are easily defined in one-dimension and easily enforced, but what the necessary and sufficient conditions for monotonicity in two and three dimensions are is less clear, see Williamson & Rasch (1989) for a discussion of one set of necessary conditions for two dimensions as put forward by Carlson & Fritsch (1985). Rather than trying to enforce a set of necessary mathematical conditions on the data to get monotonicity, which is a complicated and expensive procedure as the conditions are highly non-trivial, a more pragmatic approach has generally been taken. The simple approach, used for example by Bermejo & Staniforth (1992), is to limit the values of the interpolant to the maximum and minimum values of the data surrounding the point to which you are interpolating. For example in one dimension interpolating between points i and $i+1$ we would limit the value of the interpolant to lie between the maximum value of the quantity at i and $i+1$, and the minimum value of the quantity at i and $i+1$. In two or three dimensions there are several ways of enforcing monotonicity. For example, enforcing monotonicity on each of the one-dimensional interpolants which make up the higher order interpolant, or just limiting the final answer ? The monotone cubic Lagrange scheme we have implemented uses the latter, the monotone quasi-cubic scheme uses the former. It is possible to construct an example, given below, for terrain following co-ordinates to show that limiting each of the one-dimensional interpolants produces the correct answer, but whether there is any systematic advantage in one method over the other when used in a wider set of test problems remains to be seen.

Example. Consider stratified flow over a mountain ridge where there is no flow below the mountain top. Assuming also that the ridge has a plateau top which extends for one grid-length in the direction of the flow and that the Courant number is less than one, then the departure point for a trajectory originating at the downwind end of the ridge lies on the ridge. Interpolating the potential temperature field to the departure point should give the stratified value for that level. Using a cubic or higher order interpolant will give an overshoot when interpolating in the direction of the flow because the co-ordinate surfaces are not flat and the data values used in the interpolant vary along the surface. Limiting each of the one-dimensional interpolants gives the correct answer, but limiting just the final answer still contains an overshoot, but one which lies within the range of the maximum and minimum values of the data in the box surrounding the departure point because potential temperature is increasing with height.

3.6 Conservation

The lack of explicit conservation in semi-Lagrangian models has been a concern, particularly for climate modellers, since the method gained popularity. The problem is how to ensure conservation, in the sense that there is no change in the integral over the whole model domain of the quantity due to advection. The integral is given by the discrete form of

$$I(Q) = \frac{1}{4\pi} \int_{r_s}^{r_t} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} Q(\lambda, \phi, r)\, d\phi\, d\lambda\, dr$$

where (λ, ϕ, r) are spherical polar co-ordinates and Q is the quantity we are advecting. So far three approaches have been suggested. The first is simply to add a small correction to each point of the field to remove the error in the global integral. This a posteriori approach can be applied in many ways, and is commonly done in climate models to ensure conservation of some quantities for which explicit conservation is not possible, such as total energy. One way of enforcing conservation this way was proposed by Priestley (1993) and shown by Gravel & Staniforth (1994) to ensure mass conservation in a shallow-water model whilst having negligible impact on the solution. Other authors, for example Bates et al. (1993), have noted that the lack of formal conservation produce an error in the mass field of the order of a fraction of a hectopascal in ten days. Williamson & Olson (1989) show similar findings for the semi-Lagrangian version for the NCAR climate model and report that "there is no indication that the correction interacts with any model component to deleteriously affect the simulation". An alternative approach suggested by Leslie & Purser (1995) is to use a form of interpolation which allows direct conservation of mass and tracers. A third method suggested by Scroggs & Semazzi (1995) is to write the model in finite volume form, from whence it is relatively straight forward to enforce conservation provided that trajectories do not cross each other. In operational numerical weather prediction models forecast models it is not anticipated that enforcing conservation will have a significant impact, although Leslie & Purser (1995) have reported small positive benefits in areas of strong gradients when compared to a non-conserving schemes. However for climate simulations it is essential that conservation is enforced otherwise serious errors could occur. Currently conservation following Priestley (1993) has been implemented.

4 CONCLUSIONS

The semi-Lagrangian approach offers advantages in accuracy, stability and monotonicity that cannot be obtained from the current Eulerian schemes. The lack of formal conservation in semi-Lagrangian schemes can be addressed in several ways none of which appear to have a significant effect on the solution. Questions still remain about the optimum interpolation choice, about the calculation of values at departure points which lie outside the model domain, and about how best to enforce monotonicity. Many of these questions can only be answered by considering the performance of the alternatives in model simulations, for which the use of test problems with known physical or analytic solutions should be encouraged, such as cold gravity currents.

References

Bates, J. R. 1988 , Finite-difference semi-lagrangian techniques for integrating the shallow-water equations on the sphere, *Proc. workshop on numerical techniques for horizontal discretization in numerical weather prediction models, Reading, ECMWF* pp. 97–116.

Bates, J. R., Moorthi, S. & Higgins, R. W. 1993 , A global multilevel atmospheric model using a vector semi-lagrangian finite difference scheme. part 1: Adiabatic formulation, *Mon. Weather Rev.* **121**, 244–263.

Bermejo, R. & Staniforth, A. 1992 , The conversion of semi-lagrangian advection schemes to quasi-monotone schemes, *Mon. Wea. Rev.* **120**, 2622–2632.

Carlson, R. E. & Fritsch, F. N. 1985 , Monotone piecewise bicubic interpolation, *SIAM J. Numer. Anal.* **22**, 386–400.

Carpenter, R. L., Droegemeier, K. K., Woodward, P. R. & Hane, C. E. 1990 , Applications of the piecewise parabolic method (ppm) to meteorological modelling., *Mon. Wea. Rev.* **118**, 586–612.

Côté, J. 1988 , A lagrange multiplier approach for the metric terms of semi-lagrangian models on the sphere, *Quart. J. Roy. Met. Soc.* **114**, 1347–1352.

Cullen, M. J. P. & Davies, T. 1991 , A conservative split-explicit integration scheme with fourth-order horizontal advection, *Q. J. R. Meteorol. Soc.* **117**, 993–1002.

Cullen, M. J. P., Davies, T., Mawson, M. H., James, J. & Coulter, S. C. 1995 , An overview of numerical methods for the next generation uk nwp and climate model, To appear in Atmosphere-Ocean.

Garcia-Navarro, P. & Priestley, A. 1994 , A conservative and shape-preserving semi-lagrangian method for the solution of the shallow water equations, *Int. J. Num. Meths. in Fluids* **18**, 273–294.

Gravel, S. & Staniforth, A. 1994 , A mass conserving semi-lagrangian scheme for the shallow-water equations, *Mon. Wea. Rev.* **122**, 243–248.

Leslie, L. M. & Purser, R. J. 1995 , Three-dimensional mass-conserving semi-lagrangian scheme employing forward trajectories, *Mon. Wea. Rev.* **123**, 2551–2566.

McDonald, A. & Bates, J. R. 1987 , Improving the estimate of the departure point ina two-time level semi-lagrangian and semi-implicit scheme, *Mon. Weather Rev.* **115**, 737–739.

McGregor, J. L. 1993 , Economical determination of departure points for semi-lagrangian models, *Mon. Weather Rev.* **121**, 221–230.

Priestley, A. 1993 , A quasi-conservative version of the semi-lagrangian advection scheme, *Mon. Wea. Rev.* **121**, 621–629.

Ritchie, H. 1988 , Application of the semi-lagrangian method to a spectral model of the shallow-water equations, *Mon. Wea. Rev.* **116**, 1587–1598.

Ritchie, H. & Beaudoin, C. 1994 , Approximations and sensitivity experiments with a baroclinic semi-lagrangian spectral model, *Mon. Wea. Rev.* **122**, 2391–2399.

Ritchie, H., Temperton, C., Simmons, A., Hortal, M., Davies, T., Dent, D. & Hamrud, M. 1995 , Implementation of the semi-lagrangian method in a high resolution of the ecmwf forecast model, *Mon. Wea. Rev.* **123**, 489–514.

Robert, A. 1981 , A stable numerical integration scheme for the primitive meteorlogical equations, *Atmos. Ocean* **19**, 35–46.

Scroggs, J. S. & Semazzi, F. H. M. 1995 , A conservative semi-lagrangian method for multidimensional fluid dynamics applications, *Num. Meths. for PDEs* **11**, 445–452.

Williamson, D. L. & Olson, J. G. 1989 , Two-dimensional semi-lagrangian transport with shape-preserving interpolation, *Mon. Wea. Rev.* **117**, 102–129.

Williamson, D. L. & Rasch, P. J. 1989 , Two-dimensional semi-lagrangian transport with shape-preserving interpolation, *Mon. Wea. Rev.* **117**, 102–129.

Williamson, D. L., Drake, J. B., Hack, J. J., Jakob, R. & Swarztrauber, P. N. 1992 , A standard test set for numerical approximations to the shallow-water equations in spherical geometry, *J. Comp. Phys.* **102**, 221–224.