AN UNSTRUCTURED MESH NFT APPROACH TO ALL-SCALE ATMOSPHERIC FLOWS

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Generalised Nonhydrostatic Model

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Mass, entropy and momentum conservation laws:

Smolarkiewicz et. al. JCP 2016

$$\frac{\partial \mathcal{G}\varrho}{\partial t} + \nabla \cdot (\mathcal{G}\varrho \boldsymbol{v}) = 0$$

$$\frac{\partial \mathcal{G}\varrho\theta'}{\partial t} + \nabla \cdot (\mathcal{G}\varrho \boldsymbol{v}\theta') = -\mathcal{G}\varrho\left(\tilde{G}^T \boldsymbol{u} \cdot \nabla\theta_a - \mathcal{H}\right)$$

On a rotating sphere, cast in generalised time-dependent curvilinear coordinates, to enable a range of coordinate systems and 3D **r**-adaptivity.

$$\frac{\partial \mathcal{G}\varrho\boldsymbol{u}}{\partial t} + \nabla\cdot(\mathcal{G}\varrho\boldsymbol{v}\otimes\boldsymbol{u}) = -\mathcal{G}\varrho\left(\Theta\tilde{G}\nabla\varphi + g\Upsilon_B\frac{\theta'}{\theta_b} + \boldsymbol{f}\times(\boldsymbol{u}-\Upsilon_C\boldsymbol{u}_a) - \mathcal{M}'(\boldsymbol{u},\boldsymbol{u},\Upsilon_C) - \boldsymbol{\mathcal{D}}\right)$$

$$\varrho := \left[\rho(\boldsymbol{x},t), \frac{\rho_b(z)\theta_b(z)}{\theta(\boldsymbol{x},t)}, \rho_b(z)\right] \quad \varphi := [c_p\theta_0\pi', c_p\theta_0\pi, c_p\theta_b\pi] \quad \Theta := \left[\frac{\theta}{\theta_0}, \frac{\theta}{\theta_0}, 1\right] \quad \Upsilon_B := \left[\frac{\theta_b(z)}{\theta_a(\boldsymbol{x})}, \frac{\theta_b(z)}{\theta_a(\boldsymbol{x})}, 1\right] \quad \Upsilon_C := \left[\frac{\theta}{\theta_a(\boldsymbol{x})}, \frac{\theta}{\theta_a(\boldsymbol{x})}, 1\right]$$

$$[compressible, pseudo\text{-}incompressible, anelastic]$$

NFT Integrator

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Advection: MPDATA

Multidimensional Positive Definite
Advection Transport Algorithm

$$\frac{\partial G\Phi}{\partial t} + \nabla \cdot (\boldsymbol{V}\Phi) = G\mathcal{R}$$

$$\Phi_i^{n+1} = \mathcal{A}_i\left(\Phi^n + \tfrac{1}{2}\delta t\mathcal{R}, \boldsymbol{V}^{n+\frac{1}{2}}, G^n, G^{n+1}\right) + \tfrac{1}{2}\delta t\mathcal{R}^{n+1}$$

Smolarkiewicz & Szmelter, JCP 2009, Acta Geo 2011

Helmholtz: from evolutionary form of the gas law

$$\frac{1}{\mathcal{G}}\nabla \cdot (\mathcal{G}\boldsymbol{v}^\nu) - \frac{1}{\xi}\frac{\phi_a}{\phi^{\nu-1}}\left(\frac{1}{\varrho^*\phi_a}\nabla \cdot (\varrho^*\phi_a\boldsymbol{v}^\nu) - \frac{1}{\varrho^*}\nabla \cdot (\varrho^*\boldsymbol{v}^\nu)\right) - \frac{1}{\delta t\xi\phi^{\nu-1}}(\varphi^\nu - \hat{\varphi}) = 0$$

$$\Rightarrow \frac{1}{\zeta}\nabla \cdot \zeta\left(\check{\check{\boldsymbol{v}}} - \tilde{G}^T C\nabla\varphi\right)$$

Smolarkiewicz et. al. JCP 2016

Implicit Integrator: GCR(k)

non-symmetric, preconditioned
Krylov-subspace elliptic solver

For any initial guess, ϕ^0, set $r^0 = \mathcal{L}(\phi^0) - \mathcal{R}$, $p^0 = \mathcal{P}^{-1}(r^0)$; then iterate:

For $n = 1, 2, \ldots$ until convergence

for $\nu = 0, \ldots, k-1$

$$\beta = \frac{\langle r^\nu \mathcal{L}(p^\nu)\rangle}{\langle \mathcal{L}(p^\nu)\mathcal{L}(p^\nu)\rangle},$$

$$\phi^{\nu+1} = \phi^\nu + \beta p^\nu,$$

$$r^{\nu+1} = r^\nu + \beta\mathcal{L}(p^\nu),$$

exit if $\|r^{\nu+1}\| \le \epsilon$,

$$e = \mathcal{P}^{-1}(r^{\nu+1}),$$

evaluate $\mathcal{L}(e) = \frac{1}{\rho^*}\nabla \cdot C\nabla e$,

for $l = 0, ..\nu$

$$\alpha_l = \frac{\langle \mathcal{L}(e)\mathcal{L}(p^l)\rangle}{\langle \mathcal{L}(p^l)\mathcal{L}(p^l)\rangle},$$

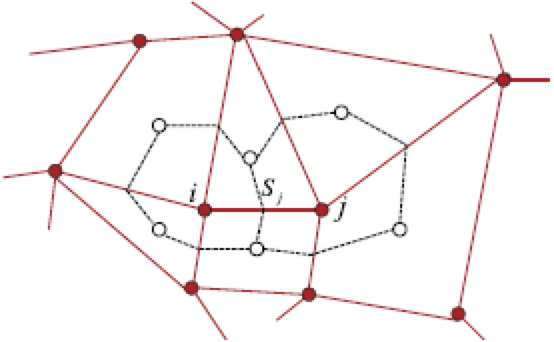
$$p^{\nu+1} = e + \textstyle\sum_l^\nu \alpha_l p^l,$$

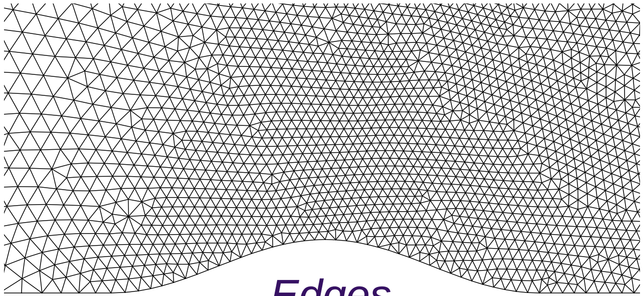
$$\mathcal{L}(p^{\nu+1}) = \mathcal{L}(e) + \textstyle\sum_l^\nu \alpha_l \mathcal{L}(e^l),$$

reset $[\phi, r, e, \mathcal{L}(e)]^k$ to $[\phi, r, e, \mathcal{L}(e)]^0$

Smolarkiewicz & Margolin, 2000

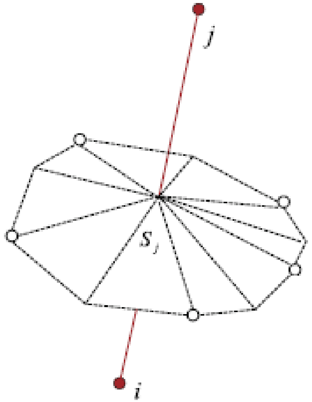
Edge based finite volume discretisation

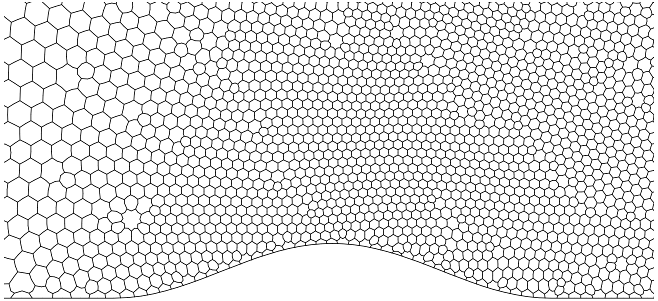




Edges

Uses a co-located data arrangement





Median dual computational mesh
Finite volumes

Smolarkiewicz & Szmelter, Acta Geo 2011

Convective Planetary Boundary Layer

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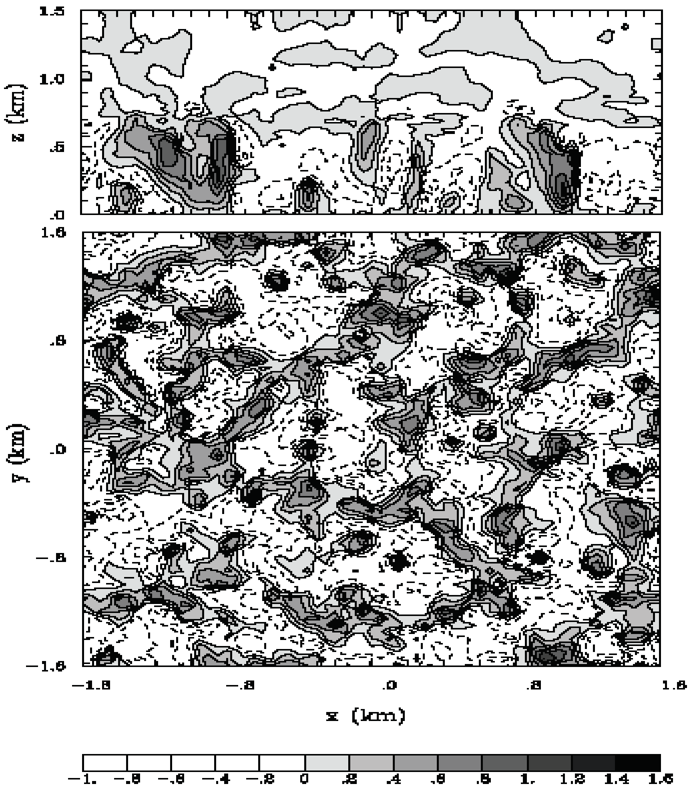
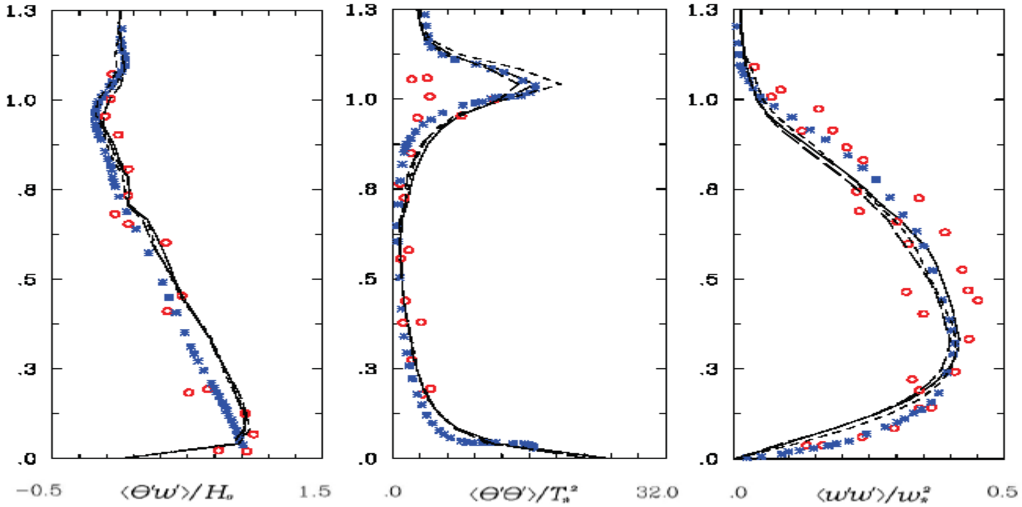
Smolarkiewicz et al JCP 2013

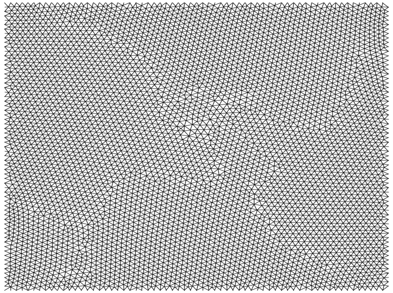
Schmidt & Schumann JFM 1989



Vertical velocity in central xz cross section (top) and the horizontal plane (bottom): instantaneous solution for triangular prismatic mesh after ~13 eddy turnover times



Vertical profile of dimensionless resolved heat flux, temperature variance, and vertical velocity; for dimensionless height z/z_i



Edge-based T

Edge-based C

EULAG

SS LES

Observation

Preconditioning

Elementary preconditioner, slower elliptic solver

Advanced Preconditioner faster elliptic solver

$$I \quad \leftarrow \quad \mathcal{P} \quad \rightarrow \quad \mathcal{L}$$

Choose $\mathcal{P}$ close to $\mathcal{L}$, but easier to solve.

Take only diagonal terms in *C*: $\mathcal{P} \approx \mathcal{L}$

Invert $\mathcal{P}_z$ implicitly, with $\mathcal{P}_H$ lagged:

$$\frac{\partial ee}{\partial \tilde{\tau}} = \mathcal{P}(e) - r \quad \Rightarrow \quad \frac{ee^{\mu+1} - ee^{\mu}}{\Delta\tilde{\tau}} = \mathcal{P}_H(e^{\mu}) + \mathcal{P}_z(e^{\mu+1}) - r$$

$$\left(I - \Delta\tilde{\tau}\mathcal{P}_z\right) e^{\mu+1} = e^{\mu} + \Delta\tilde{\tau}\left(\mathcal{P}_H(e^{\mu}) - r^{\nu+1}\right)$$

For any initial guess, ϕ^0, set $r^0 = \mathcal{L}\left(\phi^0\right) - R$, $p^0 = \mathcal{P}^{-1}\left(r^0\right)$; then iterate:

For $n = 1, 2, \ldots$ until convergence

for $\nu = 0, \ldots, k-1$

$$\beta = \frac{\langle r^{\nu}\mathcal{L}\left(p^{\nu}\right)\rangle}{\langle \mathcal{L}\left(p^{\nu}\right)\mathcal{L}\left(p^{\nu}\right)\rangle},$$

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evaluate $\mathcal{L}(e) = \dfrac{1}{\rho^*}\nabla \cdot C\nabla e$,

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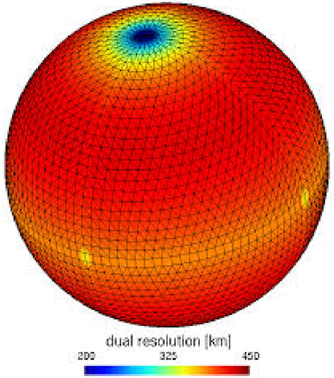
$$\alpha_l = \frac{\langle \mathcal{L}(e)\mathcal{L}\left(p^l\right)\rangle}{\langle \mathcal{L}\left(p^l\right)\mathcal{L}\left(p^l\right)\rangle},$$

$$p^{\nu+1} = e + \textstyle\sum_l^{\nu} \alpha_l p^l,$$

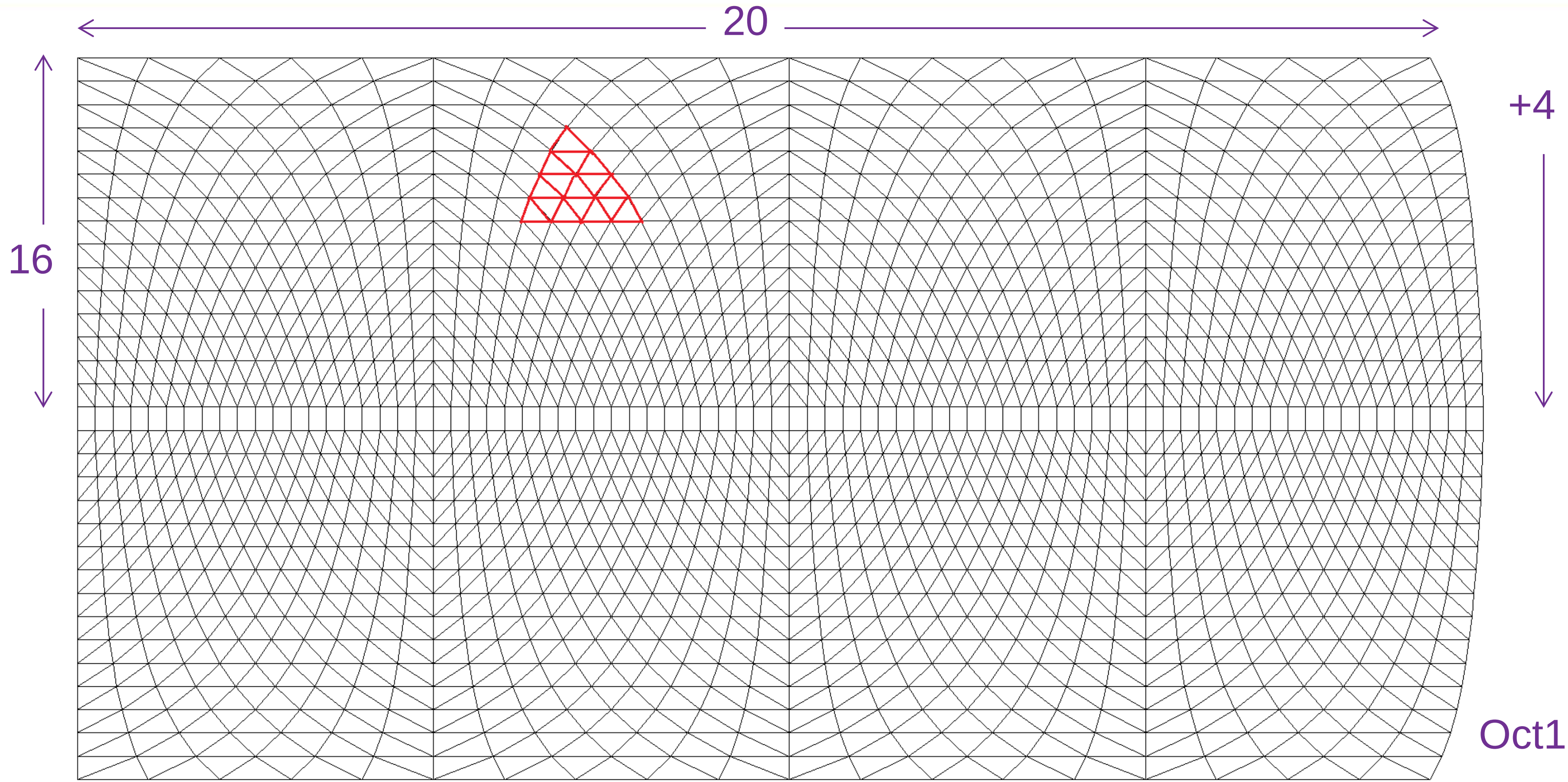
$$\mathcal{L}\left(p^{\nu+1}\right) = \mathcal{L}(e) + \textstyle\sum_l^{\nu} \alpha_l \mathcal{L}\left(e^l\right),$$

reset $[\phi, r, e, \mathcal{L}(e)]^k$ to $[\phi, r, e, \mathcal{L}(e)]^0$



Smolarkiewicz & Margolin, 2000

Multigrid meshes using Atlas



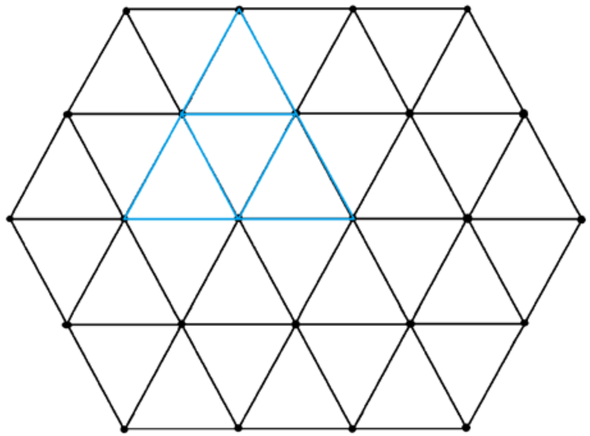
Oct16

Multigrid meshes using Atlas

Octahedral mesh has reasonably consistent structure

Latitude n = 1 . . N

has 20 + 4n longitudes



Multigrid meshes using Atlas

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Remove every other latitude

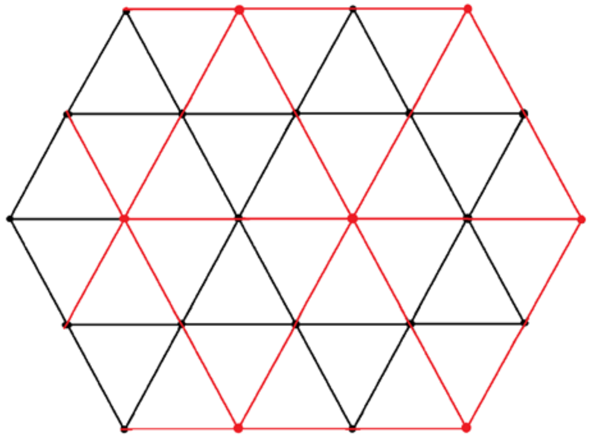
Half the number of longitudes at the remaining latitudes

Reduces horizontal nodes by a factor of ~ 4

Need to pick starting latitude

Coarse mesh is almost octahedral

All coarse mesh nodes coincide with nodes on the fine mesh



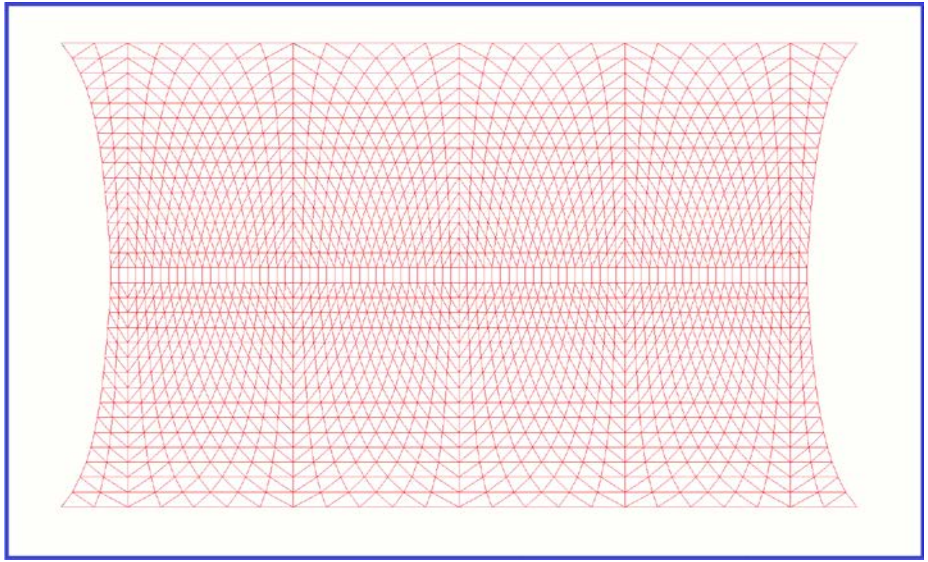
Multigrid meshes using Atlas

Octahedral 16 mesh:

computational domain

Remove odd latitudes

Single level of mesh coarsening



Multigrid meshes using Atlas

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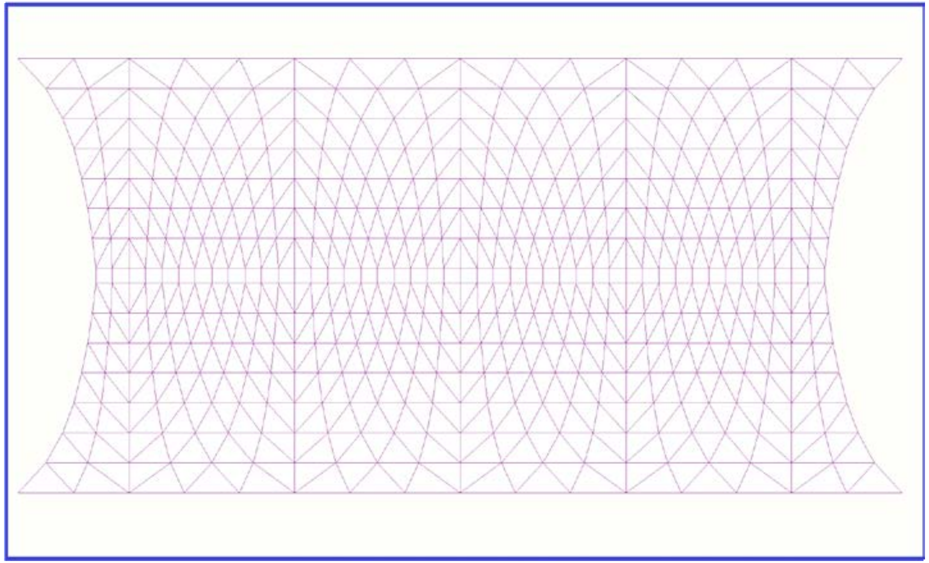
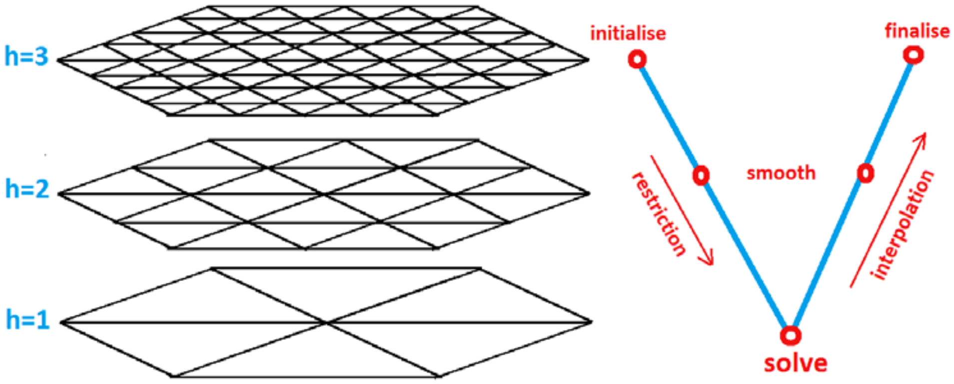


Multigrid Preconditioning

V-Cycle



Smoother / Solver, varying number of iterations depending requirements

$$\left(I - \Delta\tilde{\tau}\mathcal{P}_z\right) e^{\mu+1} = e^{\mu} + \Delta\tilde{\tau}\left(\mathcal{P}_H\left(e^{\mu}\right) - r^{\nu+1}\right)$$

V-Cycle: Solve on *h=1*, else $\mathrm{cycle}(e_h, r_h)$

$\mathrm{cycle}(e_h, r_h)$:

$\mathrm{Smooth}(e_h)$

calculate residule: $rr_h = \mathcal{P}(e_h) - r_h$

restrict $rr_h \rightarrow rr_{h-1}$: $rr_{h-1} = \mathcal{I}_h^{h-1} rr_h$

initialise solution error: $ee_{h-1}\ [\equiv e - e']$

$\mathrm{cycle}(ee_{h-1}, rr_{h-1})$

Interpolate solution error: $ee_h = \mathcal{I}_{h-1}^{h} ee_{h-1}$

Add coarse grid error: $e_h = e_h - ee_h$

$\mathrm{Smooth}(e_h)$

Multigrid Preconditioning

Baroclinic Instability – 9 days.

Compressible Nonhydrostatic Equations.

Multigrid / implicit vertical preconditioner.

Multigrid:

Wall time ~ 4hrs

Elliptic solver iterations average – 3:0

Without multigrid:

Wall time ~ 23hrs

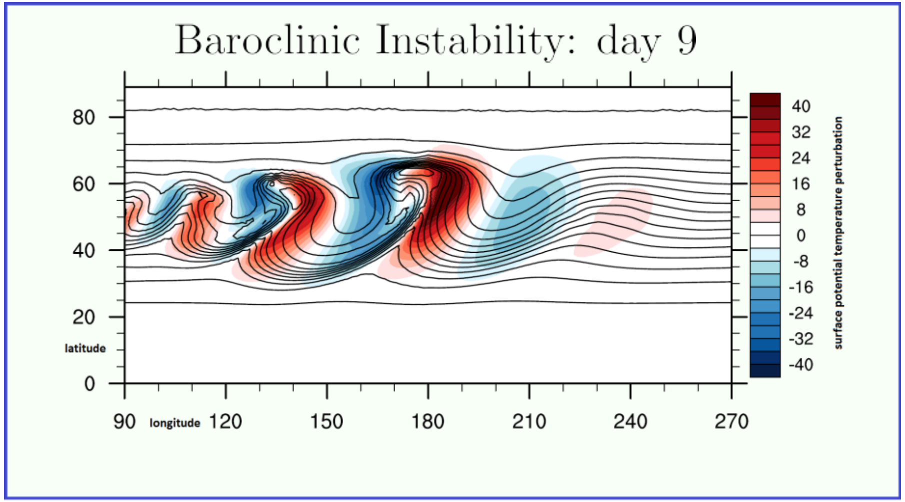
Elliptic solver iterations average – 60:7

Setup:

Octahedral 80 grid,

time-step 900s,

1/1 task/thread,



Conclusion

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- NFT-MPDATA solvers based on an edge based finite volume method for unstructured grids provide a high quality results for a wide rage of mesh types and is applicable for all scales atmospheric flows.

- Substantial efficiency gains have been achieved by introducing a horizontal multigrid preconditioner to a Krylov solver of Helmholtz equations forming a part of a global nonhydrostatic model.

ESCAPE